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Mathematical Diary 2002–2004

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Contents

Introduction

1 In the *Mathematical Diary 1993–1998*, briefly called *MD 1993–1998*, there are two different parts concerning (i) Fermat's Last Theorem (Chapters 3 and 4), and (ii) Ramsey Theory (Chapters 1,2 and 5,6,7). The part devoted to Ramsey Theory is naturally divided into two groupings of chapters namely Chapters 1 and 2 as well as Chapters 5,6, and 7.

In the first grouping Chapter 1 is devoted to a brief presentation of some basic ramseyian theorems, especially of the van der Waerden's theorem. In Chapter 2 we describe some numerical structures related to van der Waerden's theorem that possibly could lead to upper bounds of the van der Waerden numbers [After a more mature thinking we are of the opinion now (without having investigated or verified it) that the "conjectured upper bounds" in this chapter in fact are rather *not* upper bounds. The numerical data on which we were based were not sufficient for conjecturing upper bounds. Yet the question of *how these numerical structures are related to the van der Waerden's theorem and to the upper bounds of the van der Waerden numbers* remains open.].

2 In Chapter 5 of the *MD 1993–1998* we have introduced an alternative way for representing van der Waerden's theorem, a way based on enumerating all possible "colourings" of the sequence of integers. So a perspective was opened for a computational approach to the van der Waerden's theorem and especially to the van der Waerden numbers as well as to their upper bounds. In the next two chapters of the *MD 1993–1998*, namely, Chapter 6 and Chapter 7 we have presented the first steps of this computational approach "superposing the lines of $2P_j^i$'s" (these "lines" and their "superpositions" are illustrated in Figure 2 of Chapter 5) first, in Chapter 6, for constant index j and varying index i and second, in Chapter 7, for constant index i and varying index j.

3 After having completed the work appearing in the *MD 1993–1998* we attempted to continue this computational approach trying to make superpositions of the lines of $2P_j^i$'s for varying simultaneously *both* indices i and j. This kind of superposition would be more appropriate for revealing the manner by which the van der Waerden numbers and their upper bounds are formed, compared to the already examined superpositions for varying just one of the indices i and j. After many efforts, trials, experimentation, etc. we arrived at the direction for further work (direction of thought) presented here in the chapters of the *Mathematical Diary 2002–2004*, briefly called *MD 2002–2004*. We characterize it "Mathematical Diary" because (as *MD 1993–1998*) is something, regarding the style of the text, between a notebook and a book. Concerning its content *MD 2002–2004* is a direct continuation of Chapters 5,6, and 7 of the *MD 1993–1998*.

4 In Chapter 1 of the *Mathematical Diary 2002–2004* a new way for representing the lines of $2P^i_j$'s is introduced. We have come to this new way (the new style) after some investigation (and difficulty...) concerning the possibility of superposing the above lines for varying both indices i and j. This new style, as we shall see in the following chapters, facilitates greatly the superposition of the lines in a compact manner.

The new style for representing the lines of $2P^i_j$'s is mainly introduced through examples in Section 1.1 of Chapter 1. This is the most essential section of the chapter. In Section 1.2 we present more theoretically, using general formulae, what in the previous section has been presented through examples. Sections 1.1 and 1.2 are the basic content of Chapter 1. In Section 1.3*, which is of secondary importance and may be skipped (this is the meaning of the asterisk at the side of 1.3), we describe some other aspects of what is contained in Sec. 1.1 and Sec. 1.2 elucidating further the new style. Finally in Section 1.4** we try to prove a special result (a procedure) appearing in Sec. 1.3*. The content of this section is not important and because of some defects is considered "not mathematically guaranteed". So perhaps it is better for the reader to ignore it. That is why we have put two asterisks at the side of 1.4.

In the following chapters of this work only the content of Sections 1.1 and 1.2 will be used whereas the content of Sections 1.3* and 1.4** will be not met again.

5 In Chapter 2 of the *Mathematical Diary 2002–2004* we produce the form that lines $2P^i_j$ (called here on "lines $|f(i|j)|$") take when use is made of the "new style" introduced in Chapter 1.

In Section 2.1 we present some examples for the superposition of the lines $|f(i|j)|$, i.e. of the lines $2P_j^i$, with the help of the new style. From these examples it becomes clear that for proceeding further and produce the superpositions it is necessary to have previously obtained the analytical form which the lines $|f(i|j)|$ take when use of the new style is made. The production of this analytical form is the main goal of the next sections of the chapter. In Section 2.2, which is not so essential, we present some aspects of the examples and of the superpositions.

The basic part of the chapter consists of the following Sections 2.3–2.9. In these sections inductively with the help of examples we produce the analytical form of the lines $|f(i|j)|$ based on the new style. In Section 2.3 we present the examples. In Sections 2.4–2.6 we construct the general form of line $|f(i|j)|$ for the special case of $i = 1$, i.e., for $|f(1|j)|$: in Sections 2.4 and 2.5 we produce the basic units, which compose the line $|f(1|j)|$, using a recursive and a non-recursive algorithm (the basic results, in their most compact form, appear in Tables 2.5 and 2.8); in Section 2.6 we present how the basic units compose the line $|f(1|j)|$ and the main results appear in Table 2.9. In Sections 2.7–2.9 the previously derived results for line $|f(1|j)|$ are generalized for line $|f(i|j)|$. So we construct the analytical form of line $|f(i|j)|$: in Section 2.7 we produce the basic units, which compose the line $|f(i|j)|$, using a recursive algorithm and the main results appear in Table 2.10; in Section 2.8 the same units are produced through a nonrecursive algorithm and the main results, in their most compact form, appear in Table 2.14; in Section 2.9 we see how the basic units compose the line $|f(i|j)|$ and the main results appear in Table 2.16.

Thus the basic results of this chapter are presented in Tables 2.10,2.14, and 2.16. In fact the whole chapter was written exactly for the production of these results which, in the following, facilitate greatly the superposition of lines $|f(i|j)|$.

6 In Chapter 3 we present the “mechanism” by which the superposition of lines $|f(i|j)|$ takes place. For this purpose we must know the exact form that the multiples of lines $|f(i|j)|$, namely lines $2^n|f(i|j)|$, take when use of the “new style” of Chapter 1 is made. This form of lines $2^n|f(i|j)|$ is determined easily in Sections 3.1–3.5 with the help of the form of lines $|f(i|j)|$ which is determined analytically in Chapter 2.

In Sections 3.1–3.5 gradually, and with the help of examples, from partial cases we find finally the form of the general case $2^n|f(i|j)|$. The basic results appear in Tables 3.6–3.8. In Section 3.6 the form for $2^n|f(i|j)|$, given in Tables 3.6–3.8 but also in Table 3.9, is considered in detail and is written in binary style as function of exponentials of two. In Section 3.7 we describe explicitly the mechanism through which the superposition of the lines $|f(i|j)|$ is accomplished. Extended use is made of the special notation that is developed, for this purpose, in Sections 3.6 and 3.7. In the important explanatory Section 3.8 we clarify and emphasize some elementary and, essentially, already-known facts concerning the so far produced machinery. So, we shall be able to use this machinery in various applications understanding easily and without confusions what we are doing. Finally, in Section 3.9 we continue and complete the content of Section 3.7 having in mind the clarifications of Section 3.8.

Thus we are able, taking into account Chapter 3, to proceed to specific applications with respect to the superpositions of the lines $|f(i|j)|$.

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Chapter 1

A new style for representing lines $2P^{\,i}_{\,j}$ that facilitates their superposition.

1.1 Establishing the new style through examples.

1 We have passed through various attempts to superpose properly the lines of $2P^{\,i}_{\,j}$'s. Here, in order to facilitate these superpositions, we establish a new style for presenting the lines.

We introduce this new style with the help of examples.

Strings of 1 (2^0) digit:

We rename strings 0 and 1 calling them strings 1 and 2 respectively. Schematically it is

$$\underbrace{0}_{1} \quad , \quad \underbrace{1}_{2} \tag{1.1}$$

This means that the superposition table (multiplication table) for strings (digits) 0 and 1

	0	1
0	0	0
1	0	1

(1.2)

(in the table we see, for example, that string 1 superposed to string 0 gives string 0) in the new style is written

	1	2
1	1	1
2	1	2

(1.3)

Strings of 2 (2^1) digits:

We rename strings 00,01,10,11 or in the new style strings 11,12,21,22 and we call them strings 1,2,3,4 respectively. Schematically it is

$$\underbrace{00}_{1} \;,\; \underbrace{01}_{2} \;,\; \underbrace{10}_{3} \;,\; \underbrace{11}_{4} \tag{1.4}$$

or, in the new style, equivalently

$$\underbrace{11}_{1} \;,\; \underbrace{12}_{2} \;,\; \underbrace{21}_{3} \;,\; \underbrace{22}_{4} \tag{1.5}$$

The superposition table (multiplication table) for the strings 00,01,10,11 as we can easily see is

	00	01	10	11
00	00	00	00	00
01	00	01	00	01
10	00	00	10	10
11	00	01	10	11

(1.6)

Writing the strings as 11,12,21,22 the same table becomes

	11	12	21	22
11	11	11	11	11
12	11	12	11	12
21	11	11	21	21
22	11	12	21	22

(1.7)

Finally, writing the strings as 1,2,3,4 the table becomes

	1	2	3	4
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
4	1	2	3	4

(1.8)

We may write table (1.8) in a form emphasizing blocks as follows

	1	2	3	4
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
4	1	2	3	4

(1.9)

and we may do the same also for tables (1.6) and (1.7).

Strings of 4 (2^2) digits:

These strings analytically are 0000,0001,0010,$\cdots$,1111 (totally 16

strings). According to (1.4) the above strings can be written respectively 11,12,13,$\cdots$,44 [for example, string 0000 is composed of 00 and 00 which, according to (1.4), are 1 and 1 thus it is written 11, string 0001 composed of 00 and 01 i.e. of 1 and 2 is written 12, and so on].
The strings written as 11,12,13,$\cdots$,44 take new names and are called 1,2,3,$\cdots$,16 respectively. This is depicted schematically as follows

$$\begin{array}{cccc} 11 & 12 & 13 & 14 \\ 1 & 2 & 3 & 4 \\ \\ 21 & 22 & 23 & 24 \\ 5 & 6 & 7 & 8 \\ \\ 31 & 32 & 33 & 34 \\ 9 & 10 & 11 & 12 \\ \\ 41 & 42 & 43 & 44 \\ 13 & 14 & 15 & 16 \end{array} \tag{1.10}$$

Writing strings 11,12,$\cdots$,44 in their most elementary form 0000,0001,$\cdots$,1111 we present (1.10) equivalently as

$$\begin{array}{cccc} 0000 & 0001 & 0010 & 0011 \\ 1 & 2 & 3 & 4 \\ \\ 0100 & 0101 & 0110 & 0111 \\ 5 & 6 & 7 & 8 \\ \\ 1000 & 1001 & 1010 & 1011 \\ 9 & 10 & 11 & 12 \\ \\ 1100 & 1101 & 1110 & 1111 \\ 13 & 14 & 15 & 16 \end{array} \tag{1.11}$$

Next we write the superposition table for these strings. First we write the table with the strings in the form 11,12,$\cdots$,44. This is table (1.12). The sites in (1.12) are produced with the help of (1.8) as follows. Take for example the superposition of string 23 to string 41 which, according to (1.12), produces string 21. This is justified from the fact that $(23) \circ (41) = (2 \circ 4)(3 \circ 1) = 21$ since from (1.8)

	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44
11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11
12	11	12	11	12	11	12	11	12	11	12	11	12	11	12	11	12
13	11	11	13	13	11	11	13	13	11	11	13	13	11	11	13	13
14	11	12	13	14	11	12	13	14	11	12	13	14	11	12	13	14
21	11	11	11	11	21	21	21	21	11	11	11	11	21	21	21	21
22	11	12	11	12	21	22	21	22	11	12	11	12	21	22	21	22
23	11	11	13	13	21	21	23	23	11	11	13	13	21	21	23	23
24	11	12	13	14	21	22	23	24	11	12	13	14	21	22	23	24
31	11	11	11	11	11	11	11	11	31	31	31	31	31	31	31	31
32	11	12	11	12	11	12	11	12	31	32	31	32	31	32	31	32
33	11	11	13	13	11	11	13	13	31	31	33	33	31	31	33	33
34	11	12	13	14	11	12	13	14	31	32	33	34	31	32	33	34
41	11	11	11	11	21	21	21	21	31	31	31	31	41	41	41	41
42	11	12	11	12	21	22	21	22	31	32	31	32	41	42	41	42
43	11	11	13	13	21	21	23	23	31	31	33	33	41	41	43	43
44	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44

(1.12)

we have $2 \circ 4 = 2$ and $3 \circ 1 = 1$ (the symbol $\circ$ of course denotes superposition). Equivalently we may write in schematic form

$$\begin{array}{|c|c|} 2 & 3 \\ \hline 4 & 1 \end{array} = \begin{array}{|c|c|} 2 & 1 \end{array} \tag{1.13}$$

Next we write again the superposition table (1.12) now presenting the strings as 1,2,$\cdots$,16. So table (1.14) results.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2
3	1	1	3	3	1	1	3	3	1	1	3	3	1	1	3	3
4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
5	1	1	1	1	5	5	5	5	1	1	1	1	5	5	5	5
6	1	2	1	2	5	6	5	6	1	2	1	2	5	6	5	6
7	1	1	3	3	5	5	7	7	1	1	3	3	5	5	7	7
8	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8
9	1	1	1	1	1	1	1	1	9	9	9	9	9	9	9	9
10	1	2	1	2	1	2	1	2	9	10	9	10	9	10	9	10
11	1	1	3	3	1	1	3	3	9	9	11	11	9	9	11	11
12	1	2	3	4	1	2	3	4	9	10	11	12	9	10	11	12
13	1	1	1	1	5	5	5	5	9	9	9	9	13	13	13	13
14	1	2	1	2	5	6	5	6	9	10	9	10	13	14	13	14
15	1	1	3	3	5	5	7	7	9	9	11	11	13	13	15	15
16	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

(1.14)

Table (1.14) is obtained from table (1.12) just by renaming the strings according to (1.10). Of course there is no need to produce the whole table (1.12) before obtaining table (1.14). We may produce table (1.14) site-by-site with the help of (1.10) and (1.8). For example, superposition $7 \circ 14$ in (1.14) is determined analytically as follows: $7 \circ 14$ according to (1.10) is $(23) \circ (42)$ and this is $(2 \circ 4)(3 \circ 2)$ which according to (1.8) is 21 which according to (1.10) is 5; so finally we have $7 \circ 14 = 5$ and this is written in the corresponding site of (1.14). The above example schematically may be written

$$\left|\begin{array}{c} 7 \\ \hline 14 \end{array}\right| = \left|\begin{array}{c|c} 2 & 3 \\ \hline 4 & 2 \end{array}\right| = \left|\begin{array}{c|c} 2 & 1 \end{array}\right| = \left|\begin{array}{c} 5 \end{array}\right| \tag{1.15}$$

This example may equivalently be presented with the help of (1.11) and (1.2). In that case instead of e.g. (1.15) we may write

$$\left|\begin{array}{c} 7 \\ \hline 14 \end{array}\right| = \left|\begin{array}{c|c|c|c} 0 & 1 & 1 & 0 \\ \hline 1 & 1 & 0 & 1 \end{array}\right| = \left|\begin{array}{c|c|c|c} 0 & 1 & 0 & 0 \end{array}\right| = \left|\begin{array}{c} 5 \end{array}\right| \tag{1.16}$$

Here the new style, that will be used for presenting the lines $2P_j^i$, has been adequately elucidated through the examples with strings containing one, two, or four digits. The clearness of these examples will help us to avoid any possible confusion resulting from the fact that in them the same digits (integers) 1,2,3,$\cdots$ are used with different meanings.

1.2 Presenting the new style with the help of general formulae.

1 In the previous section we introduced the new style for presenting the lines $2P_j^i$ with the help of examples. Now we are going to provide general formulae for this new style as well as proofs when necessary.

As we have seen in the examples: strings of 1 (2^0) digit are of 2 species namely 1,2; strings of 2 (2^1) digits are of 4 species namely 1,2,3,4; strings of 4 (2^2) digits are of 16 species namely $1, 2, \cdots, 16$. Generally we have the following correspondence between length in digits and number of species regarding the strings

$$\begin{array}{lccccccc} \textit{length in digits:} & 2^0 & 2^1 & 2^2 & 2^3 & 2^4 & 2^5 & \cdots \\ & | & | & | & | & | & | & \cdots \\ \textit{number of species:} & 2^1 & 2^2 & 2^4 & 2^8 & 2^{16} & 2^{32} & \cdots \end{array} \tag{1.17}$$

or equivalently

$$\begin{array}{lccccccc} \textit{length in digits:} & 2^0 & 2^1 & 2^2 & 2^3 & 2^4 & 2^5 & \cdots \\ & | & | & | & | & | & | & \cdots \\ \textit{number of species:} & 2^{2^0} & 2^{2^1} & 2^{2^2} & 2^{2^3} & 2^{2^4} & 2^{2^5} & \cdots \end{array} \tag{1.18}$$

and compactly

$$\begin{array}{lc} \textit{length in digits:} & 2^{\,k} \\ & | \\ \textit{number of species:} & 2^{2^k} \end{array} \quad (k = 0, 1, 2, \cdots) \qquad (1.19)$$

Thus for any k with $k = 0, 1, 2, \cdots$ we have strings with length of $2^{\,k}$ digits and their total number (number of species) is 2^{2^k}.

2 Next we are going to repeat the procedure of Sec. 1.1 using general formulae this time.

Let us consider the case with k, that is, let us work with (superpose) strings of length $2^{\,k}$ digits ($k = 0, 1, 2, \cdots$). There are totally 2^{2^k} species (strings of this length) which analytically are $1, 2, \cdots, 2^{2^k}$. Regarding the multiplication table it is clear [since the strings finally are composed of digits 0 and 1 for which the multiplication table (1.2) holds] that: for species l and m with

$$l, m \in \{1, 2, \cdots, 2^{2^k}\} \qquad (1.20)$$

for the superposition $l \circ m$ we have

$$l \circ m \in \{1, 2, \cdots, 2^{2^k}\}. \qquad (1.21)$$

Also it is clear that $l \circ m$ is unique, and that $l \circ m = m \circ l$.

Now we consider the next case, the case for $k + 1$, and we relate this with the previous case, the case for k. In the case for $k + 1$ we have strings of length $2^{\,k+1}$ digits, and the total number of species (of strings of this length) is $2^{2^{k+1}}$. These species analytically are $1', 2', \cdots, (2^{2^{k+1}})'$. We have used accents for the species to distinguish them from the species of the previous case k. This convention will be followed hereafter: if we write species $1, 2, \cdots$ or more generally l and m we are in case k and they are chosen from set $\{1, 2, \cdots, 2^{2^k}\}$; if, on the other hand, we write species $1', 2', \cdots$ or more generally l' and m' we are in case $k + 1$ and they are chosen from set $\{1, 2, \cdots, 2^{2^{k+1}}\}$ which can also be written $\{1', 2', \cdots, (2^{2^{k+1}})'\}$. Nevertheless, the convention sometimes, for simplicity, may not be followed when all is intuitively clear and there is no danger of confusion!

Wishing to relate the species of case $k+1$ and of case k we write analytically

$$\{1', 2', \cdots, (2^{2^{k+1}})'\} =$$

$$
\begin{gathered}
\{1', 2', \cdots, ((2^{2^k})\cdot(2^{2^k}))'\} = \\
\{\ 1', 2', \cdots, (2^{2^k})', \\
\begin{array}{cccc}
(2^{2^k}+1)', & (2^{2^k}+2)', & \cdots, & (2^{2^k}+2^{2^k})' = (2\cdot 2^{2^k})', \\
(2\cdot 2^{2^k}+1)', & (2\cdot 2^{2^k}+2)', & \cdots, & (2\cdot 2^{2^k}+2^{2^k})' = (3\cdot 2^{2^k})', \\
(3\cdot 2^{2^k}+1)', & (3\cdot 2^{2^k}+2)', & \cdots, & (3\cdot 2^{2^k}+2^{2^k})' = (4\cdot 2^{2^k})', \\
\vdots & \vdots & \vdots & \vdots
\end{array} \\
((2^{2^k}-1)\cdot 2^{2^k}+1)', ((2^{2^k}-1)\cdot 2^{2^k}+2)', \cdots, ((2^{2^k}-1)\cdot 2^{2^k}+2^{2^k})' = \\
((2^{2^k})\cdot(2^{2^k}))' = (2^{2^{k+1}})'\ \}
\end{gathered}
\tag{1.22}
$$

In (1.22) we see the species (strings) for case $k+1$ enumerated one-by-one starting from number 1 (denoted $1'$) and ending with number $2^{2^{k+1}}$ [denoted $(2^{2^{k+1}})'$]. So all the elements (strings) of set $\{1, 2, \cdots, 2^{2^{k+1}}\}$ (we may add accents on the elements) are selected once each in their natural ordering. These strings are produced by concatenating in all possible combinations the strings of the previous case k which analytically are the strings (species) in the set $\{1, 2, \cdots, 2^{2^k}\}$—it is useful to remember here how, in the examples, the strings of case $k = 2$ were produced from the strings of case $k = 1$ in (1.10) [for anything we do hereafter it is elucidating to have in mind the examples of Sec. 1.1]. These concatenations of the strings of case k are

$$
\begin{array}{cccc}
11, & 12, & \cdots, & 1(2^{2^k}), \\
21, & 22, & \cdots, & 2(2^{2^k}), \\
31, & 32, & \cdots, & 3(2^{2^k}), \\
41, & 42, & \cdots, & 4(2^{2^k}), \\
\vdots & \vdots & \vdots & \vdots \\
(2^{2^k})1, & (2^{2^k})2, & \cdots, & (2^{2^k})(2^{2^k})
\end{array}
\tag{1.23}
$$

Taking into account (1.10),(1.22), and (1.23) it is clear that the strings appearing in the third part of Eq. (1.22) are exactly those (are equal to those) appearing in (1.23) in one-to-one correspondence in the order in which they appear.

Note 1.1 Writing the strings (species) we must emphasize the difference between *products of numbers* and *concatenations of strings*. For example in (1.22) there is $(2^{2^k}) \cdot (2^{2^k})$ and $2 \cdot 2^{2^k}$ and $3 \cdot 2^{2^k}$ etc. which are products of numbers, whereas in (1.23) there is $(2^{2^k})(2^{2^k})$ which is concatenation of string 2^{2^k} and the same string 2^{2^k}, there is 31 which is concatenation of string 3 and string 1, there is $4(2^{2^k})$ which is concatenation of string 4 and string 2^{2^k}, and so on. This difference must be kept in mind during the whole remaining part of the present chapter! □

3 Thus from (1.22) and (1.23) [having also in mind the example (1.10)] we obtain directly the equations

$$(0 \cdot 2^{2^k} + \lambda)' = 1\lambda \quad \text{with} \quad \lambda \in \{1, 2, \cdots, 2^{2^k}\} \tag{1.24a}$$

$$(1 \cdot 2^{2^k} + \lambda)' = 2\lambda \quad \text{with} \quad \lambda \in \{1, 2, \cdots, 2^{2^k}\} \tag{1.24b}$$

$$(2 \cdot 2^{2^k} + \lambda)' = 3\lambda \quad \text{with} \quad \lambda \in \{1, 2, \cdots, 2^{2^k}\} \tag{1.24c}$$

$$(3 \cdot 2^{2^k} + \lambda)' = 4\lambda \quad \text{with} \quad \lambda \in \{1, 2, \cdots, 2^{2^k}\} \tag{1.24d}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$((2^{2^k} - 1) \cdot 2^{2^k} + \lambda)' = \underbrace{2^{2^k}\lambda}_{\text{concatenations}} \quad \text{with} \quad \lambda \in \{1, 2, \cdots, 2^{2^k}\} \tag{1.24e}$$

Writing Eqs. (1.24) more compactly they become

$$[\,(\mu - 1) \cdot 2^{2^k} + \lambda\,]' = \mu\lambda$$
$$\text{with} \quad \lambda \in \{1, 2, \cdots, 2^{2^k}\} \quad \text{and} \quad \mu \in \{1, 2, \cdots, 2^{2^k}\} \tag{1.25}$$

[The values of μ and λ in (1.25) are given in such way that numbers $[\,(\mu - 1) \cdot 2^{2^k} + \lambda\,]$ are cited in their natural increasing order and strings $\mu\lambda$ are cited in lexicographic order, see (1.10),(1.22),(1.23), and (1.24).]

Let us take strings l' and m' such that

$$l', m' \in \{1', 2', \cdots, (2^{2^{k+1}})'\}. \tag{1.26}$$

Strings of this kind, according to (1.22),(1.24), and (1.25), can be written in the form of the first part of the equation in (1.25) thus

$$\begin{aligned} l' &= [(\mu_1 - 1)\cdot 2^{2^k} + \lambda_1]' \\ m' &= [(\mu_2 - 1)\cdot 2^{2^k} + \lambda_2]' \end{aligned} \tag{1.27}$$

with $\mu_1, \lambda_1, \mu_2, \lambda_2 \in \{1, 2, \cdots, 2^{2^k}\}$. So there is a one-to-one correspondence between l' and m' on the one side and the pairs (μ_1, λ_1) and (μ_2, λ_2) on the other side respectively. Schematically it is

$$\begin{aligned} l' &\leftrightarrow (\mu_1, \lambda_1) \\ m' &\leftrightarrow (\mu_2, \lambda_2) \end{aligned} \tag{1.28}$$

4 It is possible now to write a general formula producing the superposition $l' \circ m'$ for any l' and m' if the superpositions $l \circ m$ for any l and m, with $l, m \in \{1, 2, \cdots, 2^{2^k}\}$, are known.

This formula is (1.29).

$$l' \circ m' = [(\mu_1 - 1)\cdot 2^{2^k} + \lambda_1]' \circ [(\mu_2 - 1)\cdot 2^{2^k} + \lambda_2]' =$$

$$\underbrace{(\mu_1\lambda_1)}_{\text{concatenation}} \circ \underbrace{(\mu_2\lambda_2)}_{\text{concatenation}} = \underbrace{(\mu_1 \circ \mu_2)(\lambda_1 \circ \lambda_2)}_{\text{concatenation}} =$$

$$[((\mu_1 \circ \mu_2) - 1)\cdot 2^{2^k} + (\lambda_1 \circ \lambda_2)]' \tag{1.29}$$

(If we know any $l \circ m$ then we know $\mu_1 \circ \mu_2$ and $\lambda_1 \circ \lambda_2$ thus we determine any $l' \circ m'$)

Let us see an example for the formula (1.29). For case $k = 1$, thus $k + 1 = 2$, according to our notation and to (1.20) and (1.26) it is $l, m \in \{1, 2, 3, 4\}$ and $l', m' \in \{1', 2', \cdots, (16)'\}$. Provided that we know any product $l \circ m$ [which means that we have at our disposal the multiplication table (1.8)] let us find $l' \circ m'$ for $l' = 7'$ and $m' = (11)'$ [which is equal to $3'$ as we know from the multiplication table (1.14)] with the help of (1.29).

Writing (1.27) and (1.28) for $(l', m') = (7', (11)')$ we have

$$\begin{aligned} 7' &= [(2 - 1)\cdot 2^{2^1} + 3]' \\ (11)' &= [(3 - 1)\cdot 2^{2^1} + 3]' \end{aligned} \tag{1.30}$$

and

$$\begin{array}{rcll} 7' & = & l' & \leftrightarrow \ (\mu_1, \lambda_1) = (2,3) \\ (11)' & = & m' & \leftrightarrow \ (\mu_2, \lambda_2) = (3,3) \end{array} \qquad (1.31)$$

respectively. Thus from (1.29) we obtain directly

$$7' \circ (11)' = [\,(2-1)\cdot 2^{2^1} + 3\,]' \circ [\,(3-1)\cdot 2^{2^1} + 3\,]' =$$

$$(23) \circ (33) = (2 \circ 3)(3 \circ 3) =$$

$$[\,((2 \circ 3) - 1)\cdot 2^{2^1} + (3 \circ 3)\,]'. \qquad (1.32)$$

And, since from table (1.8) we have $2 \circ 3 = 1$ and $3 \circ 3 = 3$, Eq. (1.32) finally becomes

$$7' \circ (11)' = [\,(1-1)\cdot 2^{2^1} + 3\,]' = 3' \qquad (1.33)$$

which is exactly what we obtain from table (1.14)!

5 We have seen now with the help of general formulae how from the superposition table for the strings of case k we obtain the superposition table for the strings of the next case $k+1$. The first step of the procedure is for $k = 0$ when, according to the examples in Sec. 1.1 [see (1.1)–(1.3)], the strings have number of digits 2^0 i.e. 1 and their total number (number of species) is 2^{2^0} i.e. 2, see also (1.19). These strings are the ones in the set $\{1, 2\}$ and their multiplication table is (1.3). The content of the table, analytically written, is that for strings $l, m \in \{1, 2\}$ it is $l \circ m \in \{1, 2\}$ and that $1 \circ 1 = 1 \circ 2 = 2 \circ 1 = 1$, and $2 \circ 2 = 2$.

6 Let us "play" for a while with the structure of the multiplication tables. Specifically we are going to investigate this structure with respect to the "sub-blocks" of the tables. First we shall *present compactly* the structure regarding *the subblocks* and next *a proof* for this structure will be given with the help of the algorithm of (1.29).

The structure of the subblocks.

Let us consider the multiplication table for case k. This is a matrix (block) with side of length 2^{2^k} digits (sites), i.e., it is a matrix $2^{2^k} \times 2^{2^k}$. If $l, m \in \{1, 2, \cdots, 2^{2^k}\}$ then $l \circ m$ is a site in the matrix. When considering site $l \circ m$ in the matrix of case k we may say that the site has *coordinates* l and m as well as that the site has *numerical value* $l \circ m$.

Now let us examine how is the form of the subblock (of dimension $2^{2^k} \times 2^{2^k}$) with "coordinates" l and m in the matrix of $k+1$ [indeed, as we can see easily in the examples, each site with coordinates l and m in the matrix of k corresponds to a subblock of the matrix of $k+1$ with same dimension as the matrix of k; so the subblock with dimension $2^{2^k} \times 2^{2^k}$, in the matrix of $k+1$ (which has dimension $2^{2^{k+1}} \times 2^{2^{k+1}}$), which corresponds to site $l \circ m$ of the matrix of k may be characterized as "the subblock with coordinates l and m" of the matrix of $k+1$]. We can easily see in the examples of Sec. 1.1 (and we prove in the following) that in the matrix of $k+1$ for the subblock (of dimension $2^{2^k} \times 2^{2^k}$) with coordinates l and m there are the following features characterizing its structure: a) each site of this subblock is in one-to-one correspondence with a site of the matrix of k; b) the numerical value of the site (p, q) of the subblock (with $p, q \in \{1, 2, \cdots, 2^{2^k}\}$), which we denote as $v'(p, q)$ [symbol v is taken from the word "value"], is related to the numerical value of the corresponding site (p, q) of the matrix of k, which we denote as $v(p, q)$ [and we know it is $v(p, q) = p \circ q$], by the formula

$$v'(p,q) = [\,(l \circ m) - 1\,] \cdot 2^{2^k} + v(p,q) \tag{1.34}$$

The proof of Eq. (1.34).

We consider in the matrix of $k+1$ the subblock with "coordinates" l and m. In this subblock we choose a site which in the subblock has coordinates (p, q). This site in the matrix of $k+1$ has coordinates l' and m' for which, as we can easily see, we have the relation

$$(l', m') = [\,(l-1) \cdot 2^{2^k} + p\,,\ (m-1) \cdot 2^{2^k} + q\,]\,. \tag{1.35}$$

Alternatively we may write (1.35) in the form

$$(l', m') = [\,[\,(l-1) \cdot 2^{2^k} + p\,]'\,,\ [\,(m-1) \cdot 2^{2^k} + q\,]'\,]\,. \tag{1.36}$$

Comparing (1.36) with (1.27) we obtain

$$(\mu_1, \lambda_1, \mu_2, \lambda_2) = (l, p, m, q)\,. \tag{1.37}$$

Since the site (l', m') of the matrix of $k+1$ and the site (p, q) of the subblock, with "coordinates" l and m, of the matrix (block) of $k+1$ are one and the same site they have the same numerical value. So it is

$$v'(p,q) = l' \circ m' \,. \tag{1.38}$$

Combining (1.38),(1.29), and (1.37) we obtain

$$v'(p,q) = l' \circ m' =$$

$$[\,(\mu_1 - 1) \cdot 2^{2^k} + \lambda_1\,]' \circ [\,(\mu_2 - 1) \cdot 2^{2^k} + \lambda_2\,]' =$$

$$[\,((\mu_1 \circ \mu_2) - 1) \cdot 2^{2^k} + (\lambda_1 \circ \lambda_2)\,]' =$$

$$[\,((l \circ m) - 1) \cdot 2^{2^k} + (p \circ q)\,]'. \tag{1.39}$$

And since $v(p,q) = p \circ q$ we obtain from (1.39)

$$v'(p,q) = [\,((l \circ m) - 1) \cdot 2^{2^k} + v(p,q)\,]'. \tag{1.40}$$

Since the second part of Eq. (1.40) may equivalently be written without the accent Eq. (1.40) is exactly Eq. (1.34). Thus the proof is complete. □

With the help of Eq. (1.34) we can from the matrix of k produce the matrix of $k+1$ by constructing each subblock separately. That is, we have a recursive procedure that starts from the case of $k = 0$ (first step of the recursion), when the matrix is (1.3), and continues for any k using (1.34).

7 The structure described in Section 1.2 ,Subsection 6 (we may write Sec. 1.2 ,Subsec. 6 or just Sec. 1.2 ,6), especially in formula (1.34), it is good to be clarified through proper illustrations. So we present the following.

Matrix for (of) k:

[the matrix has dimension $(2^{2^k} \times 2^{2^k})$]

$l \diagdown m$	1	2	$\cdots$	2^{2^k}
1	α_{11}	α_{12}	$\cdots$	$\alpha_{12^{2^k}}$
2	α_{21}	α_{22}	$\cdots$	$\alpha_{22^{2^k}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{2^k}	$\alpha_{2^{2^k}1}$	$\alpha_{2^{2^k}2}$	$\cdots$	$\alpha_{2^{2^k}2^{2^k}}$

$$= (\alpha_{lm})_{l,m \in \{1,2,\cdots,2^{2^k}\}} \tag{1.41}$$

with

$$\alpha_{lm} = l \circ m \quad \text{and} \quad \alpha_{lm} \in \{1, 2, \cdots, 2^{2^k}\} \tag{1.42}$$

Matrix for (of) $k+1$:

[the matrix has dimension $(2^{2^{k+1}} \times 2^{2^{k+1}})$]

$l \diagdown m$	1	2	$\cdots$	2^{2^k}
1	A_{11}	A_{12}	$\cdots$	$A_{12^{2^k}}$
2	A_{21}	A_{22}	$\cdots$	$A_{22^{2^k}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{2^k}	$A_{2^{2^k}1}$	$A_{2^{2^k}2}$	$\cdots$	$A_{2^{2^k}2^{2^k}}$

$$= (A_{lm})_{l,m \in \{1,2,\cdots,2^{2^k}\}} \tag{1.43}$$

with A_{lm} a matrix given by

$$A_{lm} =$$

$p \diagdown q$	1	2	$\cdots$	2^{2^k}
1	α'_{11}	α'_{12}	$\cdots$	$\alpha'_{12^{2^k}}$
2	α'_{21}	α'_{22}	$\cdots$	$\alpha'_{22^{2^k}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{2^k}	$\alpha'_{2^{2^k}1}$	$\alpha'_{2^{2^k}2}$	$\cdots$	$\alpha'_{2^{2^k}2^{2^k}}$

$$= (\alpha'_{pq})_{p,q \in \{1,2,\cdots,2^{2^k}\}} \tag{1.44}$$

and where

$$\alpha'_{pq} = (\alpha_{lm} - 1) \cdot 2^{2^k} + \alpha_{pq} \quad \text{and} \quad \alpha'_{pq} \in \{1, 2, \cdots, 2^{2^{k+1}}\} \tag{1.45}$$

8 Repeating more compactly the essential parts of Sec. 1.2,Subsec. 7 we have

Matrix for k:

$$(\alpha_{lm})_{l,m\in\{1,2,\cdots,2^{2^k}\}} \tag{1.46}$$

with

$$\alpha_{lm} \in \{1,2,\cdots,2^{2^k}\} \tag{1.47}$$

Matrix for $k+1$:

$$(A_{lm})_{l,m\in\{1,2,\cdots,2^{2^k}\}} \tag{1.48}$$

with

$$A_{lm} = ({\alpha'}_{pq})_{p,q\in\{1,2,\cdots,2^{2^k}\}} \tag{1.49}$$

and with

$${\alpha'}_{pq} \in \{1,2,\cdots,2^{2^{k+1}}\} \quad \text{and} \quad {\alpha'}_{pq} = (\alpha_{lm}-1)\cdot 2^{2^k} + \alpha_{pq} \tag{1.50}$$

And as first step of the recursive procedure we have

Matrix for $k=0$:

(dimension 2×2) (1.51)

	1	2
1	1	1
2	1	2

which is

$$(\alpha_{lm})_{l,m\in\{1,2\}} \tag{1.52}$$

with $\alpha_{11}=\alpha_{12}=\alpha_{21}=1$, and $\alpha_{22}=2$ [it is the matrix (1.3)].

9 No special proof for the results in Sec. 1.2 ,7 and Sec. 1.2 ,8 is needed since proof is the content of Sec. 1.2 ,6 [especially the part *The proof of Eq. (1.34)*]!

10 Let us see the tables of Sec. 1.1 as examples of the “matrix-procedure” of Sec. 1.2 ,7 and Sec. 1.2 ,8.

For $k = 0$:

The table (1.3) is just the first step of the recursive procedure described in Sec. 1.2 ,7 and Sec. 1.2 ,8 [see (1.41)–(1.52)] and it is the matrix (1.51) i.e.

1	1
1	2

(1.53)

For $k = 1$:

Considering matrix (1.53) as the matrix (1.41) for $k = 0$ we easily see that matrix (1.43) for $k + 1 = 1$ is just matrix (1.8) or (1.9). Indeed, matrix (1.8) or (1.9) may be analysed as

1	1	1	1
1	2	1	2
1	1	3	3
1	2	3	4

=

0·2+1	0·2+1	0·2+1	0·2+1
0·2+1	0·2+2	0·2+1	0·2+2
0·2+1	0·2+1	1·2+1	1·2+1
0·2+1	0·2+2	1·2+1	1·2+2

(1.54)

In the 4×4 matrix of (1.54) the 2×2 subblocks (which are clearly depicted) are just the submatrices A_{lm} determined through (1.44) and (1.45).

For $k = 2$:

Considering the matrix in (1.54) as the matrix (1.41) for $k = 1$ we can see that matrix (1.43) for $k + 1 = 2$ is matrix (1.14). The 4×4 subblocks of the 16×16 matrix (1.14) are the submatrices A_{lm} which can be determined through (1.44) and (1.45). Because of the big size of matrix (1.14) we shall not analyse it in the style of the previous example of (1.54). Instead, in that style we analyse below only the four basic 4×4 submatrices of (1.14) and, with the help of (1.44) and (1.45), we observe how they are related to matrix (1.8) [which is the first part of (1.54)].

The 4×4 submatrix of (1.14) for which $(l', m') \in \{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$:

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 1 & 2 & 1 & 2 \\ \hline 1 & 1 & 3 & 3 \\ \hline 1 & 2 & 3 & 4 \\ \hline \end{array} = \begin{array}{|c|c|c|c|}\hline 0{\cdot}4{+}1 & 0{\cdot}4{+}1 & 0{\cdot}4{+}1 & 0{\cdot}4{+}1 \\ \hline 0{\cdot}4{+}1 & 0{\cdot}4{+}2 & 0{\cdot}4{+}1 & 0{\cdot}4{+}2 \\ \hline 0{\cdot}4{+}1 & 0{\cdot}4{+}1 & 0{\cdot}4{+}3 & 0{\cdot}4{+}3 \\ \hline 0{\cdot}4{+}1 & 0{\cdot}4{+}2 & 0{\cdot}4{+}3 & 0{\cdot}4{+}4 \\ \hline \end{array} \tag{1.55}$$

The 4×4 *submatrix of (1.14) for which* $(l', m') \in \{5, 6, 7, 8\} \times \{5, 6, 7, 8\}$*:*

$$\begin{array}{|c|c|c|c|}\hline 5 & 5 & 5 & 5 \\ \hline 5 & 6 & 5 & 6 \\ \hline 5 & 5 & 7 & 7 \\ \hline 5 & 6 & 7 & 8 \\ \hline \end{array} = \begin{array}{|c|c|c|c|}\hline 1{\cdot}4{+}1 & 1{\cdot}4{+}1 & 1{\cdot}4{+}1 & 1{\cdot}4{+}1 \\ \hline 1{\cdot}4{+}1 & 1{\cdot}4{+}2 & 1{\cdot}4{+}1 & 1{\cdot}4{+}2 \\ \hline 1{\cdot}4{+}1 & 1{\cdot}4{+}1 & 1{\cdot}4{+}3 & 1{\cdot}4{+}3 \\ \hline 1{\cdot}4{+}1 & 1{\cdot}4{+}2 & 1{\cdot}4{+}3 & 1{\cdot}4{+}4 \\ \hline \end{array} \tag{1.56}$$

The 4×4 *submatrix of (1.14) for which* $(l', m') \in \{9, 10, 11, 12\} \times \{9, 10, 11, 12\}$*:*

$$\begin{array}{|c|c|c|c|}\hline 9 & 9 & 9 & 9 \\ \hline 9 & 10 & 9 & 10 \\ \hline 9 & 9 & 11 & 11 \\ \hline 9 & 10 & 11 & 12 \\ \hline \end{array} = \begin{array}{|c|c|c|c|}\hline 2{\cdot}4{+}1 & 2{\cdot}4{+}1 & 2{\cdot}4{+}1 & 2{\cdot}4{+}1 \\ \hline 2{\cdot}4{+}1 & 2{\cdot}4{+}2 & 2{\cdot}4{+}1 & 2{\cdot}4{+}2 \\ \hline 2{\cdot}4{+}1 & 2{\cdot}4{+}1 & 2{\cdot}4{+}3 & 2{\cdot}4{+}3 \\ \hline 2{\cdot}4{+}1 & 2{\cdot}4{+}2 & 2{\cdot}4{+}3 & 2{\cdot}4{+}4 \\ \hline \end{array} \tag{1.57}$$

The 4×4 *submatrix of (1.14) for which* $(l', m') \in \{13, 14, 15, 16\} \times \{13, 14, 15, 16\}$*:*

$$\begin{array}{|c|c|c|c|}\hline 13 & 13 & 13 & 13 \\ \hline 13 & 14 & 13 & 14 \\ \hline 13 & 13 & 15 & 15 \\ \hline 13 & 14 & 15 & 16 \\ \hline \end{array} = \begin{array}{|c|c|c|c|}\hline 3{\cdot}4{+}1 & 3{\cdot}4{+}1 & 3{\cdot}4{+}1 & 3{\cdot}4{+}1 \\ \hline 3{\cdot}4{+}1 & 3{\cdot}4{+}2 & 3{\cdot}4{+}1 & 3{\cdot}4{+}2 \\ \hline 3{\cdot}4{+}1 & 3{\cdot}4{+}1 & 3{\cdot}4{+}3 & 3{\cdot}4{+}3 \\ \hline 3{\cdot}4{+}1 & 3{\cdot}4{+}2 & 3{\cdot}4{+}3 & 3{\cdot}4{+}4 \\ \hline \end{array} \tag{1.58}$$

It is clear from (1.55)–(1.58) that the sites inside the 4×4 sub-blocks result by adding multiples of 4 to the corresponding sites of matrix (1.8) in accordance with the procedure of (1.43)–(1.45).

1.3* Elucidating the new style further.

[The content of this section (marked with an asterisk) is not essential for what follows in the next chapters so it may be skipped.]

Now we present the structure of the matrices of Sec. 1.1 *in different ways*, compared to the way of Sec. 1.2,6, wishing to make it more clear.

1 It is possible to present the structure of the matrices of Sec. 1.1 in the following way (which is rather intuitively clear and we expose it without proof).

We have the matrix for the case of k. As we have seen in the part *The structure of the subblocks* of Sec. 1.2,6 the side is of length 2^{2^k} digits (sites), i.e., it is a $2^{2^k} \times 2^{2^k}$ matrix.

Let us choose a random site (l, m) with $l, m \in \{1, 2, \cdots, 2^{2^k}\}$ and numerical value $l \circ m$.

Now we deal with the matrix of $k + 1$ which, as we know, has side of length $2^{2^{k+1}}$ digits thus being a $2^{2^{k+1}} \times 2^{2^{k+1}}$ matrix.

In this matrix we consider the four sites

$$(l, m), (l, m + 2^{2^k}), (l + 2^{2^k}, m), (l + 2^{2^k}, m + 2^{2^k}) \qquad (1.59)$$

[we call this: "step 1"]. The coordinates l, m are exactly as in the matrix of k. Then:

$$l \circ m \quad \text{is as in the matrix of } k \qquad (1.60a)$$

$$l \circ (m + 2^{2^k}) = (l + 2^{2^k}) \circ m = l \circ m \qquad (1.60b)$$

$$(l + 2^{2^k}) \circ (m + 2^{2^k}) = l \circ m + 2^{2^k} \qquad (1.60c)$$

Next we proceed to step 2 for constructing the matrix of $k + 1$: We choose a random site (l', m') in the submatrix, of the matrix of $k+1$, with sides $(1, 2, \cdots, 2 \cdot 2^{2^k}) \times (1, 2, \cdots, 2 \cdot 2^{2^k})$ or equivalently $(1, 2, \cdots, 2^{2^k+1}) \times (1, 2, \cdots, 2^{2^k+1})$. It is $l', m' \in \{1, 2, \cdots, 2^{2^k+1}\}$. Then we consider the four sites

$$(l', m'), (l', m' + 2^{2^k+1}), (l' + 2^{2^k+1}, m'), (l' + 2^{2^k+1}, m' + 2^{2^k+1}) \quad (1.61)$$

for which, similarly, we have:

$$l' \circ m' \quad \text{is known since the submatrix is known} \qquad (1.62a)$$

$$l' \circ (m' + 2^{2^k+1}) = (l' + 2^{2^k+1}) \circ m' = l' \circ m' \tag{1.62b}$$

$$(l' + 2^{2^k+1}) \circ (m' + 2^{2^k+1}) = l' \circ m' + 2^{2^k+1} \tag{1.62c}$$

Next, similarly, we proceed to step 3 for constructing the matrix of $k+1$: We choose a new random site (l'', m'') in the submatrix, of matrix $k+1$, with sides $(1, 2, \cdots, 2 \cdot 2^{2^k+1}) \times (1, 2, \cdots, 2 \cdot 2^{2^k+1})$ or $(1, 2, \cdots, 2^{2^k+2}) \times (1, 2, \cdots, 2^{2^k+2})$. It is $l'', m'' \in \{1, 2, \cdots, 2^{2^k+2}\}$. Then we consider the four sites

$$(l'', m''), (l'', m'' + 2^{2^k+2}), (l'' + 2^{2^k+2}, m''), (l'' + 2^{2^k+2}, m'' + 2^{2^k+2}) \tag{1.63}$$

for which we have:

$$l'' \circ m'' \quad \text{is known from the known submatrix} \tag{1.64a}$$

$$l'' \circ (m'' + 2^{2^k+2}) = (l'' + 2^{2^k+2}) \circ m'' = l'' \circ m'' \tag{1.64b}$$

$$(l'' + 2^{2^k+2}) \circ (m'' + 2^{2^k+2}) = l'' \circ m'' + 2^{2^k+2} \tag{1.64c}$$

We continue in the same manner until finally we arrive at (the final) step 2^k: We choose a new random site (l^a, m^a) in the submatrix, of matrix $k+1$, with sides $(1, 2, \cdots, 2 \cdot 2^{2^k+(2^k-2)}) \times (1, 2, \cdots, 2 \cdot 2^{2^k+(2^k-2)})$ or $(1, 2, \cdots, 2^{2^k+(2^k-1)}) \times (1, 2, \cdots, 2^{2^k+(2^k-1)})$. It is $l^a, m^a \in \{1, 2, \cdots, 2^{2^k+(2^k-1)}\}$. Then we consider the four sites

$$\begin{aligned}&(l^a, m^a), (l^a, m^a + 2^{2^k+(2^k-1)}), (l^a + 2^{2^k+(2^k-1)}, m^a),\\ &\qquad\qquad (l^a + 2^{2^k+(2^k-1)}, m^a + 2^{2^k+(2^k-1)})\end{aligned} \tag{1.65}$$

for which we have:

$$l^a \circ m^a \quad \text{known} \tag{1.66a}$$

$$l^a \circ (m^a + 2^{2^k+(2^k-1)}) = (l^a + 2^{2^k+(2^k-1)}) \circ m^a = l^a \circ m^a \tag{1.66b}$$

$$(l^a + 2^{2^k+(2^k-1)}) \circ (m^a + 2^{2^k+(2^k-1)}) = l^a \circ m^a + 2^{2^k+(2^k-1)} \tag{1.66c}$$

Thus the matrix of $k+1$ with dimension $2^{2^{k+1}}$ [it is a $(2^{2^{k+1}}) \times (2^{2^{k+1}})$-matrix] has been completely constructed.

We have presented here a procedure by which from the matrix of k we construct step-by-step the matrix of $k+1$. The procedure makes the structure of these matrices clear.

2 Now the above described procedure is presented more compactly.

We start from the matrix of k (which is a known matrix) with sites (l, m) and values $l \circ m$, where $l, m \in \{1, 2, \cdots, 2^{2^k}\}$. Step-by-step we construct matrix $k+1$. Totally we have 2^k steps. Denoting each step by "ν" (with $\nu \in \{1, 2, \cdots, 2^k\}$) to each such step there corresponds a "constructed-by-the-procedure" matrix: for $\nu = 1$ we construct the next matrix to the matrix of k, for $\nu = 2$ the next matrix to the previous one, and so on; finally for $\nu = 2^k$ we constuct the matrix of $k+1$. For the random step ν we start from a matrix of dimension $2^{2^k+(\nu-1)}$ [a $(2^{2^k+(\nu-1)} \times 2^{2^k+(\nu-1)})$-matrix] and we construct a matrix of dimension $2^{2^k+\nu}$ [a $(2^{2^k+\nu} \times 2^{2^k+\nu})$ -matrix] with the following property:

For any coordinates l' and m' of the new matrix with

$$l', m' \in \{1, 2, \cdots, 2^{2^k+(\nu-1)}\}, \tag{1.67}$$

for the four sites

$$\begin{gathered}(l', m'), (l', m' + 2^{2^k+(\nu-1)}), (l' + 2^{2^k+(\nu-1)}, m'), \\ (l' + 2^{2^k+(\nu-1)}, m' + 2^{2^k+(\nu-1)})\end{gathered} \tag{1.68}$$

we have

$$l' \circ m' \quad \text{known (as in the old matrix)} \tag{1.69a}$$

$$l' \circ (m' + 2^{2^k+(\nu-1)}) = (l' + 2^{2^k+(\nu-1)}) \circ m' = l' \circ m' \tag{1.69b}$$

$$(l' + 2^{2^k+(\nu-1)}) \circ (m' + 2^{2^k+(\nu-1)}) = l' \circ m' + 2^{2^k+(\nu-1)} \tag{1.69c}$$

3 Illustratively we can show the procedure as follows:

Step ν:

Old matrix: $A \quad (2^{2^k+(\nu-1)} \times 2^{2^k+(\nu-1)})$

New matrix:

$$\begin{array}{|c|c|}\hline A & A \\ \hline A & A' \\ \hline\end{array} \qquad (2^{2^k+\nu} \times 2^{2^k+\nu}) \tag{1.70}$$

Each site of A' (i.e., its value) is equal to the corresponding site (i.e., to its value) of A plus $2^{2^k+(\nu-1)}$.

For the first step $\nu = 1$ of case $k = 0$ the old matrix is the (2×2)-matrix (1.53); so the procedure starts and continues...

4 We are not going to provide a proof concerning the above described procedure. But, wishing to see what such a proof would seem like and also wishing to be more familiar with some aspects of all these things, we offer a proof of (1.69) for some partial case.

In Sec. 1.3*,2 we have seen compactly how from matrix k we construct matrix $k+1$ through steps ν.

Let us present a proof of (1.69) for the general k and step $\nu = 1$.

The old matrix is matrix k. The new matrix is the part of matrix $k+1$ that is defined by the coordinates

$$(1, 2, \cdots, 2^{2^k+1}) \times (1, 2, \cdots, 2^{2^k+1}). \tag{1.71}$$

So for (l', m'), which is a site of matrix $k+1$, we have from (1.29):

$$\begin{aligned} l' \circ m' &= [\,(\mu_1 - 1) \cdot 2^{2^k} + \lambda_1\,]' \circ [\,(\mu_2 - 1) \cdot 2^{2^k} + \lambda_2\,]' \\ &= [\,((\mu_1 \circ \mu_2) - 1) \cdot 2^{2^k} + (\lambda_1 \circ \lambda_2)\,]' \end{aligned} \tag{1.72}$$

Since it is $\mu_1 = 1,\ \lambda_1 = l$ and $\mu_2 = 1,\ \lambda_2 = m$ we have:

$$l' \circ m' = [\,((1 \circ 1) - 1) \cdot 2^{2^k} + (l \circ m)\,]' = (l \circ m)' \tag{1.73}$$

Having arrived at (1.73) the proof of (1.69a) is complete (provided that we can prove the relation $1 \circ 1 = 1$).

Next we prove (1.69b). Using again (1.29) we have:

$$\begin{aligned} & l' \circ (m' + 2^{2^k}) = l' \circ (m' + 2^{2^k})' \\ & = [\,(\mu_1 - 1) \cdot 2^{2^k} + \lambda_1\,]' \circ [\,(\mu_2 - 1) \cdot 2^{2^k} + \lambda_2\,]' \\ & = [\,((\mu_1 \circ \mu_2) - 1) \cdot 2^{2^k} + (\lambda_1 \circ \lambda_2)\,]' \end{aligned} \tag{1.74}$$

Since it is $\mu_1 = 1,\ \lambda_1 = l$ and $\mu_2 = 2,\ \lambda_2 = m$, and also taking into account (1.73), the last part of (1.74) becomes successively:

$$[\,((1 \circ 2) - 1) \cdot 2^{2^k} + (l \circ m)\,]' = (l \circ m)' = l' \circ m' \tag{1.75}$$

Now, combining (1.74) with (1.75), the one part of (1.69b) has been proved (provided that we can prove $1 \circ 2 = 1$).

Next we prove the other part of (1.69b). Using similarly (1.29) we obtain:

$$\begin{aligned} &(l' + 2^{2^k}) \circ m' = (l' + 2^{2^k})' \circ m' \\ &= [\,(\mu_1 - 1) \cdot 2^{2^k} + \lambda_1\,]' \circ [\,(\mu_2 - 1) \cdot 2^{2^k} + \lambda_2\,]' \\ &= [\,((\mu_1 \circ \mu_2) - 1) \cdot 2^{2^k} + (\lambda_1 \circ \lambda_2)\,]' \end{aligned} \tag{1.76}$$

Since it is $\mu_1 = 2,\ \lambda_1 = l$ and $\mu_2 = 1,\ \lambda_2 = m$, and also taking into account (1.73), the last part of (1.76) becomes successively:

$$[\,((2 \circ 1) - 1) \cdot 2^{2^k} + (l \circ m)\,]' = (l \circ m)' = l' \circ m' \tag{1.77}$$

Now, combining (1.76) with (1.77), the other part of (1.69b) has been proved as well (provided that we can prove $2 \circ 1 = 1$). So the whole (1.69b) has been proved.

Finally we prove (1.69c). As before using (1.29) we have:

$$\begin{aligned} &(l' + 2^{2^k}) \circ (m' + 2^{2^k}) = (l' + 2^{2^k})' \circ (m' + 2^{2^k})' \\ &= [\,(\mu_1 - 1) \cdot 2^{2^k} + \lambda_1\,]' \circ [\,(\mu_2 - 1) \cdot 2^{2^k} + \lambda_2\,]' \\ &= [\,((\mu_1 \circ \mu_2) - 1) \cdot 2^{2^k} + (\lambda_1 \circ \lambda_2)\,]' \end{aligned} \tag{1.78}$$

Since it is $\mu_1 = 2,\ \lambda_1 = l$ and $\mu_2 = 2,\ \lambda_2 = m$, and also taking into account (1.73), the last part of (1.78) becomes successively:

$$[\,((2 \circ 2) - 1) \cdot 2^{2^k} + (l \circ m)\,]' = (2^{2^k} + l' \circ m')' = l' \circ m' + 2^{2^k} \tag{1.79}$$

So, combining (1.78) with (1.79), (1.69c) has been proved (provided that we can prove $2 \circ 2 = 2$) and this means that the proof of (1.69), for the general k and for step $\nu = 1$, is complete.

5 Let us see the features of the procedure, described in Sec. 1.3*,2 and Sec. 1.3*,3 [especially in (1.70)], from another standpoint.

We know from (1.29) how matrix $k + 1$ is related to matrix k. Considering these matrices we see that matrix $k+1$ has length of side $2^{2^{k+1}}$ sites, whereas matrix k has length of side 2^{2^k} sites. Doubling

2^{2^k} to cover $2^{2^{k+1}}$ we have 2^k doublings $[\,2 \cdot 2^{2^k} = 2^{2^k+1},\ 2 \cdot 2^{2^k+1} = 2^{2^k+2}, \cdots, 2 \cdot 2^{2^k+(2^k-1)} = 2^{2^k+2^k} = 2^{2^{k+1}}\,]$, each such doubling corresponding to a step ν of Sec. 1.3*,2 [so $\nu \in \{1, 2, \cdots, 2^k\}$]. Let us take a random doubling, i.e., a random step ν. At this step we have (with the consecutive doublings) constructed the submatrix of matrix $k+1$ that has sides with coordinates $(1, 2, \cdots, 2^{2^k+\nu}) \times (1, 2, \cdots, 2^{2^k+\nu})$. This submatrix is divided into four further submatrices I,II,III, and IV, i.e., into those submatrices having sides with coordinates

$$(1, 2, \cdots, 2^{2^k+(\nu-1)}) \times (1, 2, \cdots, 2^{2^k+(\nu-1)}) \quad \text{(submatrix I)} \quad (1.80a)$$

$$\begin{gathered}(1, 2, \cdots, 2^{2^k+(\nu-1)}) \times \\ (2^{2^k+(\nu-1)} + 1, 2^{2^k+(\nu-1)} + 2, \cdots, 2^{2^k+\nu}) \quad \text{(submatrix II)}\end{gathered} \quad (1.80b)$$

$$\begin{gathered}(2^{2^k+(\nu-1)} + 1, 2^{2^k+(\nu-1)} + 2, \cdots, 2^{2^k+\nu}) \times \\ (1, 2, \cdots, 2^{2^k+(\nu-1)}) \quad \text{(submatrix III)}\end{gathered} \quad (1.80c)$$

$$\begin{gathered}(2^{2^k+(\nu-1)} + 1, 2^{2^k+(\nu-1)} + 2, \cdots, 2^{2^k+\nu}) \times \\ (2^{2^k+(\nu-1)} + 1, 2^{2^k+(\nu-1)} + 2, \cdots, 2^{2^k+\nu}) \quad \text{(submatrix IV)}\end{gathered} \quad (1.80d)$$

We can choose a random site (l', m') in submatrix I and then we may consider its corresponding, according to (1.68), sites in submatrices II,III, and IV. We may also consider the relations (1.69).

6 In (1.27) and (1.28) we see how l' and m' are related to $\mu_1, \lambda_1, \mu_2, \lambda_2$. In (1.27) l' and m' are expressed as functions of $\mu_1, \lambda_1, \mu_2, \lambda_2$. Here we express $\mu_1, \lambda_1, \mu_2, \lambda_2$ as functions of l' and m'. In fact it is enough to consider only l' since for m' we work similarly.

Having in mind (1.28) we may, instead of μ_1, λ_1 corresponding to l' and μ_2, λ_2 corresponding to m', write $\mu(l'), \lambda(l')$ and $\mu(m'), \lambda(m')$ respectively.

So from (1.27) we have

$$l' = [\,(\mu(l') - 1) \cdot 2^{2^k} + \lambda(l')\,]', \tag{1.81}$$

with $l' \in \{1, 2, \cdots, 2^{2^{k+1}}\}$ and $\mu, \lambda \in \{1, 2, \cdots, 2^{2^k}\}$. Now the following hold for $\mu(l')$ and $\lambda(l')$.

If $rem(l'/2^{2^k}) \in \{1, 2, \cdots, 2^{2^k} - 1\}$

then

$$\begin{aligned} \mu(l') &= quot(l'/2^{2^k}) + 1, \\ \lambda(l') &= rem(l'/2^{2^k}). \end{aligned} \tag{1.82}$$

If $rem(l'/2^{2^k}) = 0$ [i.e., if $l' = \alpha \cdot 2^{2^k}$ with $\alpha \in \{1, 2, \cdots, 2^{2^k}\}$]

then

$$\begin{aligned} \mu(l') &= quot(l'/2^{2^k}), \\ \lambda(l') &= 2^{2^k}. \end{aligned} \tag{1.83}$$

7 In Sec. 1.3*,1–3 we have presented a procedure revealing the basic features of the structure of the matrix k. Here those features are presented in another alternative manner.

We consider matrix k according to its description in Sec. 1.2,7. This matrix has side of 2^{2^k} sites. We take a submatrix of matrix k in the following way. For $\nu \in \{1, 2, \cdots, 2^k\}$ we consider respectively sides of length $2^1, 2^2, \cdots, 2^{2^k}$ sites, that is, generally of length 2^ν sites. So for some ν we have the illustration of matrix k and of the submatrix appearing in Fig. 1.2.

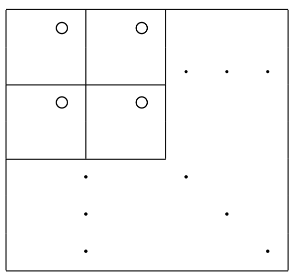

Fig. 1.2 Matrix and submatrices.

In Fig. 1.2 there are: four squares of minimal size representing submatrices with length of side $2^{\nu-1}$ sites each; a square of intermediary size, composed of the previous four squares, representing a submatrix with length of side 2^ν sites; the maximal square, containing all the others, representing matrix k with length of side 2^{2^k} sites; four points (sites), appearing in the figure as small circles. Each of the four points has been posed in one of the four minimal squares and at exactly the same place of each square. So these four points (sites), for random $l, m \in \{1, 2, \cdots, 2^{\nu-1}\} \subseteq \{1, 2, \cdots, 2^{2^k}\}$, analytically are:

$$(l, m), (l, m + 2^{\nu-1}), (l + 2^{\nu-1}, m), (l + 2^{\nu-1}, m + 2^{\nu-1}) \qquad (1.84)$$

The four points appearing in (1.84) are those which in Fig. 1.2 are depicted up-left, up-right, down-left, down-right respectively as we can directly verify. In, e.g., point (l, m) by "coordinate" l we denote the distance of the point from the upper side of the maximal square (matrix k) of Fig. 1.2, whereas by "coordinate" m we denote the distance of the point from the left side of the maximal square. Similarly for the coordinates of the other points in (1.84).

For the four points (sites) of (1.84) and Fig. 1.2 we *accept* intuitively (and try to prove formally next) the following property:

$$l \circ (m + 2^{\nu-1}) = (l + 2^{\nu-1}) \circ m = l \circ m \,, \qquad (1.85a)$$

and

$$(l + 2^{\nu-1}) \circ (m + 2^{\nu-1}) = l \circ m + 2^{\nu-1} \,. \qquad (1.85b)$$

8 Property (1.85) is evidently true for matrix $k = 0$. Suppose that this property holds for a matrix of random k (for any $\nu \in \{1, 2, \cdots, 2^k\}$). Then if we could prove the property for matrix $k+1$ (for any $\nu \in \{1, 2, \cdots, 2^{k+1}\}$) that would be an inductive proof of the property for any k.

Let us write analytically this property for matrix $k + 1$.

Matrix $k + 1$ has side of $2^{2^{k+1}}$ sites (which is equal to $2^{2^k} \cdot 2^{2^k}$). Working exactly as for matrix k in Sec. 1.3*,7 we take a submatrix of matrix $k + 1$ as follows. For $\nu \in \{1, 2, \cdots, 2^{k+1}\}$ we consider respectively sides of length $2^1, 2^2, \cdots, 2^{2^{k+1}}$ sites, that is, generally of length 2^ν sites. So for some ν we have, for matrix $k + 1$ and for the submatrix, again the illustration in Fig. 1.2 but now with some differences.

Specifically, the maximal square here represents matrix $k+1$ and has side with length $2^{2^{k+1}}$ sites; also for the four points (sites) we have random $l', m' \in \{1, 2, \cdots, 2^{\nu-1}\} \subseteq \{1, 2, \cdots, 2^{2^{k+1}}\}$ and these analytically are:

$$(l', m'), (l', m' + 2^{\nu-1}), (l' + 2^{\nu-1}, m'), (l' + 2^{\nu-1}, m' + 2^{\nu-1}) \quad (1.86)$$

Now if we prove the relations

$$l' \circ (m' + 2^{\nu-1}) = (l' + 2^{\nu-1}) \circ m' = l' \circ m' \,, \quad (1.87a)$$

$$(l' + 2^{\nu-1}) \circ (m' + 2^{\nu-1}) = l' \circ m' + 2^{\nu-1} \,, \quad (1.87b)$$

which are the corresponding to (1.85) for matrix $k+1$, we have also proved by induction (1.85) for any matrix k.

1.4** Trying to prove (1.87) and thus to prove (1.85) for any matrix *k*.

[The content of this section (marked with two asterisks) is totally unrelated to what follows, not especially important, somehow messy, and (above all) *not guaranteed* for mathematical correctness! We have included it because it offers an experience of how we work with all these things. Perhaps there would be better for the reader to ignore it.]

1 Something characterizing what is contained in this chapter is that things intuitively clear may require a lot of work when are formally presented or formally proved. So properties and procedures transparent from the examples perhaps it is not always necessary to acquire formal descriptions and proofs. The property in (1.85) may be intuitively understandable but here we attempt to find a formal proof. That is, we wish to find a proof for (1.87) and thus to prove (1.85) for any k. During this effort we see how the machinery established so far can be used in practical applications.

Something like a proof of (1.87).

2 We emphasize again that the only difference between Fig. 1.2 for matrix k and the same figure for matrix $k+1$ is that: the maximal square is matrix k with side of length 2^{2^k} sites in Fig. 1.2 for matrix

k, whereas the maximal square is matrix $k+1$ with side of length $2^{2^{k+1}}$ sites in Fig. 1.2 for matrix $k+1$; the other squares are same and with same length of side in Fig. 1.2 for matrix k and in Fig. 1.2 for matrix $k+1$ but in the figure of matrix k (we say briefly "in the figure of k") for the ν contained in the length of sides we have $\nu \in \{1, 2, \cdots, 2^k\}$, whereas in the figure of matrix $k+1$ (we say "in the figure of $k+1$") for ν we have $\nu \in \{1, 2, \cdots, 2^{k+1}\}$; also in Fig. 1.2 for k there are l and m with $l, m \in \{1, 2, \cdots, 2^{\nu-1}\} \subseteq \{1, 2, \cdots, 2^{2^k}\}$, whereas in Fig. 1.2 for $k+1$ there are l' and m' with $l', m' \in \{1, 2, \cdots, 2^{\nu-1}\} \subseteq \{1, 2, \cdots, 2^{2^{k+1}}\}$. All the other features of Fig. 1.2 for k and of Fig. 1.2 for $k+1$ are same.

3 For $\nu \in \{1, 2, \cdots, 2^k\}$ it is $max(\nu) = 2^k$ so $max(2^\nu) = 2^{max(\nu)} = 2^{2^k}$ and this means that the maximal possible submatrix of matrix $k+1$ with side 2^ν (have in mind Fig. 1.2 for $k+1$) is just matrix k (have in mind Fig. 1.2 for k) which has side 2^{2^k}. Taking into account Sec. 1.2,7 we can see that really this submatrix of matrix $k+1$ is exactly matrix k [since in matrix k it is $\alpha_{11} = 1$ $(1 \circ 1 = 1)$ we have $\alpha'_{pq} = (\alpha_{lm} - 1) \cdot 2^{2^k} + \alpha_{pq} = (1-1) \cdot 2^{2^k} + \alpha_{pq} = \alpha_{pq}$ and so A_{11} is same as matrix k]. So the four points in Fig. 1.2 for $k+1$, i.e. points (1.86), are just as the four points in Fig. 1.2 for k, i.e. points (1.84). This means that $l' = l$ and $m' = m$ hence relations (1.87) are just relations (1.85). This is what we wished to prove for the case under consideration with $\nu \in \{1, 2, \cdots, 2^k\}$.

Next we consider the case with $\nu \in \{2^k+1, 2^k+2, \cdots, 2^{k+1}\}$.

4 With the help of Sec. 1.2,7 we see that

$$l' \circ (m' + 2^{\nu-1}) = \alpha'_{p(l')q(m'+2^{\nu-1})} =$$

$$(\alpha_{l(l')m(m'+2^{\nu-1})} - 1) \cdot 2^{2^k} + \alpha_{p(l')q(m'+2^{\nu-1})}. \tag{1.88}$$

Having in mind Sec. 1.1 (the examples) and Sec. 1.2 [especially the part between (1.22) and (1.29)] we can see (perhaps...) that the following relations hold:

$$l(l') = \mu(l'), \qquad m(m' + 2^{\nu-1}) = \mu(m' + 2^{\nu-1}), \tag{1.89a}$$

and

$$l(l') = \mu(l'), \qquad m(m') = \mu(m'), \tag{1.89b}$$

also

$$p(l') = \lambda(l'), \qquad q(m' + 2^{\nu-1}) = \lambda(m' + 2^{\nu-1}), \tag{1.89c}$$

and

$$p(l') = \lambda(l'), \qquad q(m') = \lambda(m'). \tag{1.89d}$$

Using (1.89) the last part of (1.88) becomes

$$(\alpha_{\mu(l')\mu(m'+2^{\nu-1})} - 1) \cdot 2^{2^k} + \alpha_{\lambda(l')\lambda(m'+2^{\nu-1})}. \tag{1.90}$$

Also it is

$$l' \circ m' = \alpha'_{p(l')q(m')} = (\alpha_{l(l')m(m')} - 1) \cdot 2^{2^k} + \alpha_{p(l')q(m')}$$
$$= (\alpha_{\mu(l')\mu(m')} - 1) \cdot 2^{2^k} + \alpha_{\lambda(l')\lambda(m')}. \tag{1.91}$$

So for being

$$l' \circ (m' + 2^{\nu-1}) = l' \circ m' \tag{1.92}$$

it is enough to prove that

$$(\alpha_{\mu(l')\mu(m'+2^{\nu-1})} - 1) \cdot 2^{2^k} + \alpha_{\lambda(l')\lambda(m'+2^{\nu-1})}$$
$$= (\alpha_{\mu(l')\mu(m')} - 1) \cdot 2^{2^k} + \alpha_{\lambda(l')\lambda(m')}. \tag{1.93}$$

Given the range of α, the above relation can be true if and only if

$$\alpha_{\mu(l')\mu(m'+2^{\nu-1})} = \alpha_{\mu(l')\mu(m')}, \tag{1.94a}$$

and

$$\alpha_{\lambda(l')\lambda(m'+2^{\nu-1})} = \alpha_{\lambda(l')\lambda(m')}. \tag{1.94b}$$

5 For proving (1.94a) and (1.94b) we take into account Sec. 1.2,7 as well as the definitions in Sec. 1.3*,6. Let us see it analytically.

We can deduce directly from Sec. 1.2,7 that

$$\lambda(m' + 2^{\nu-1}) = \lambda(m'). \tag{1.95}$$

We can obtain (1.95) also analytically from the definitions of Sec. 1.3*,6 [i.e. of (1.82) and (1.83)]:

— if $m' \neq \alpha \cdot 2^{2^k}$ with $\alpha \in \{1, 2, \cdots, 2^{2^k}\}$ then, since

$$\nu \in \{2^k + 1, 2^k + 2, \cdots, 2^{k+1}\}, \tag{1.96}$$

it is also

$$m' + 2^{\nu-1} \neq \alpha \cdot 2^{2^k}, \tag{1.97}$$

hence

$$rem(m'/2^{2^k}), rem((m' + 2^{\nu-1})/2^{2^k}) \in \{1, 2, \cdots, 2^{2^k} - 1\}, \tag{1.98}$$

and so

$$\mu(m') = quot(m'/2^{2^k}) + 1, \tag{1.99a}$$

$$\lambda(m') = rem(m'/2^{2^k}), \tag{1.99b}$$

and

$$\mu(m' + 2^{\nu-1}) = quot((m' + 2^{\nu-1})/2^{2^k}) + 1, \tag{1.100a}$$

$$\lambda(m' + 2^{\nu-1}) = rem((m' + 2^{\nu-1})/2^{2^k}); \tag{1.100b}$$

— also if $m' = \alpha \cdot 2^{2^k}$ then it is also

$$m' + 2^{\nu-1} = \alpha \cdot 2^{2^k}, \tag{1.101}$$

hence

$$rem(m'/2^{2^k}) = rem((m' + 2^{\nu-1})/2^{2^k}) = 0, \tag{1.102}$$

and so

$$\mu(m') = quot(m'/2^{2^k}), \tag{1.103a}$$

$$\lambda(m') = 2^{2^k}, \tag{1.103b}$$

and

$$\mu(m' + 2^{\nu-1}) = quot((m' + 2^{\nu-1})/2^{2^k}), \tag{1.104a}$$

$$\lambda(m' + 2^{\nu-1}) = 2^{2^k}. \tag{1.104b}$$

So if $m' \neq \alpha \cdot 2^{2^k}$, given that $\nu \in \{2^k + 1, 2^k + 2, \cdots, 2^{k+1}\}$, it is $rem((m' + 2^{\nu-1})/2^{2^k}) = rem(m'/2^{2^k})$. So because of (1.99b) and (1.100b) we obtain $\lambda(m' + 2^{\nu-1}) = \lambda(m')$, that is, we obtain (1.95). Also if $m' = \alpha \cdot 2^{2^k}$ then $\lambda(m' + 2^{\nu-1}) = 2^{2^k} = \lambda(m')$ because of (1.103b) and (1.104b), so also we obtain (1.95).

Thus Eq. (1.95) is always true hence Eq. (1.94b) is also always true!

6 Next, let us prove Eq. (1.94a) intuitively from Sec. 1.2,7 and systematically from Sec. 1.3*,6 [i.e. from Eqs. (1.82) and (1.83) that previously produced Eqs. (1.99),(1.100),(1.103),(1.104)].

From Sec. 1.2,7 intuitively we see directly the validity of Eq. (1.94a). Let us see it systematically.

Eq. (1.94a) equivalently is written

$$[\,\mu(l')\,] \circ [\,\mu(m' + 2^{\nu-1})\,] = [\,\mu(l')\,] \circ [\,\mu(m')\,]. \tag{1.105}$$

Eq. (1.105) refers to matrix k so we wish to reduce it to (1.85a).

It is $l', m' \in \{1, 2, \cdots, 2^{\nu-1}\}$ (have in mind Fig. 1.2 for $k+1$) for $\nu \in \{2^k + 1, 2^k + 2, \cdots, 2^{k+1}\}$.

It is better in (1.105) to write $\nu\,'$ instead of ν with $\nu\,' \in \{2^k + 1, 2^k + 2, \cdots, 2^{k+1}\}$; $\nu\,'$ refers to matrix $k+1$. Then for ν, that refers to matrix k, it is $\nu \in \{1, 2, \cdots, 2^k\}$ so we have

$$\nu\,' = 2^k + \nu. \tag{1.106}$$

Thus (1.105) is written

$$[\,\mu(l')\,] \circ [\,\mu(m' + 2^{\nu\,'-1})\,] = [\,\mu(l')\,] \circ [\,\mu(m')\,]. \tag{1.107}$$

$$\text{If} \quad l' \neq \alpha \cdot 2^{2^k} \quad \text{it is} \quad \mu(l') = quot(l'/2^{2^k}) + 1 \tag{1.108a}$$

$$\text{If} \quad l' = \alpha \cdot 2^{2^k} \quad \text{it is} \quad \mu(l') = quot(l'/2^{2^k}) \tag{1.108b}$$

Since $l' \in \{1, 2, \cdots, 2^{\nu'-1}\}$ with $\nu' \in \{2^k+1, 2^k+2, \cdots, 2^{k+1}\}$ and $\nu' = 2^k + \nu$, it is $l' \in \{1, 2, \cdots, 2^{2^k+\nu-1}\}$. So for $l' = 2^{2^k+\nu-1}$ since $l' = \alpha \cdot 2^{2^k}$ it is $\mu(l') = quot(l'/2^{2^k}) = quot(2^{2^k+\nu-1}/2^{2^k}) = 2^{\nu-1}$, and for $l' = 2^{2^k+\nu-1} - 1$ since $l' \neq \alpha \cdot 2^{2^k}$ it is $\mu(l') = quot(l'/2^{2^k}) + 1 = quot((2^{2^k+\nu-1} - 1)/2^{2^k}) + 1 = (2^{\nu-1} - 1) + 1 = 2^{\nu-1}$; so it is $\mu(l') \in \{1, 2, \cdots, 2^{\nu-1}\} \subseteq \{1, 2, \cdots, 2^{2^k}\}$ thus $\mu(l')$ is as l of (1.84) and (1.85) and we can put

$$\underline{\mu(l') = l} \in \{1, 2, \cdots, 2^{\nu-1}\} \subseteq \{1, 2, \cdots, 2^{2^k}\} \tag{1.109}$$

with $\nu \in \{1, 2, \cdots, 2^k\}$.

Now, if $m' \neq \alpha \cdot 2^{2^k}$ it is also $m' + 2^{\nu'-1} \neq \alpha \cdot 2^{2^k}$ hence $\mu(m') = quot(m'/2^{2^k}) + 1$, and $\mu(m' + 2^{\nu'-1}) = quot((m' + 2^{\nu'-1})/2^{2^k}) + 1$; if $m' = \alpha \cdot 2^{2^k}$ it is also $m' + 2^{\nu'-1} = \alpha \cdot 2^{2^k}$ hence $\mu(m') = quot(m'/2^{2^k})$, and $\mu(m' + 2^{\nu'-1}) = quot((m' + 2^{\nu'-1})/2^{2^k})$. Since $m' \in \{1, 2, \cdots, 2^{\nu'-1}\}$ and $\nu' \in \{2^k+1, 2^k+2, \cdots, 2^{k+1}\}$ and $\nu' = 2^k + \nu$, it is $m' \in \{1, 2, \cdots, 2^{2^k+\nu-1}\}$. So working exactly as before (i.e., for l') we obtain finally that $\mu(m')$ is as m of (1.84) and (1.85) and we can put

$$\underline{\mu(m') = m} \in \{1, 2, \cdots, 2^{\nu-1}\} \subseteq \{1, 2, \cdots, 2^{2^k}\} \tag{1.110}$$

with $\nu \in \{1, 2, \cdots, 2^k\}$.

As for $m' + 2^{\nu'-1}$:

— if $m' \neq \alpha \cdot 2^{2^k}$ and $m' + 2^{\nu'-1} \neq \alpha \cdot 2^{2^k}$ then we have

$$\begin{aligned} \mu(m' + 2^{\nu'-1}) &= quot((m' + 2^{\nu'-1})/2^{2^k}) + 1 \\ &= quot((m' + 2^{2^k+\nu-1})/2^{2^k}) + 1 \\ = quot(m'/2^{2^k}) + 2^{\nu-1} + 1 &= quot(m'/2^{2^k}) + 1 + 2^{\nu-1} \\ &= \mu(m') + 2^{\nu-1} = m + 2^{\nu-1}; \end{aligned} \tag{1.111}$$

— if $m' = \alpha \cdot 2^{2^k}$ and $m' + 2^{\nu'-1} = \alpha \cdot 2^{2^k}$ then we have

$$\mu(m' + 2^{\nu'-1}) = quot((m' + 2^{\nu'-1})/2^{2^k})$$

$$= quot((m' + 2^{2^k+\nu-1})/2^{2^k})$$

$$= quot(m'/2^{2^k}) + 2^{\nu-1} = \mu(m') + 2^{\nu-1} = m + 2^{\nu-1}. \tag{1.112}$$

So always it is

$$\underline{\mu(m' + 2^{\nu'-1}) = m + 2^{\nu-1}} \tag{1.113}$$

with m and ν as in (1.110).

So from (1.109),(1.110), and (1.113) equation (1.107) becomes

$$l \circ (m + 2^{\nu-1}) = l \circ m, \tag{1.114}$$

which is true as we know from (1.85a)!

Thus we have proved (1.94a). Since we have already proved (1.94b) this means that we have also proved part $l' \circ (m' + 2^{\nu-1}) = l' \circ m'$ of (1.87a).

7 Now we prove part $(l' + 2^{\nu-1}) \circ m' = l' \circ m'$ of (1.87a). It is better to write this equation using ν' instead of ν with $\nu' \in \{2^k + 1, 2^k + 2, \cdots, 2^{k+1}\}$. We work exactly as for the previous equation $l' \circ (m' + 2^{\nu-1}) = l' \circ m'$. No need to repeat the procedure. Since always it is $l \circ m = m \circ l$, we may prove equation

$$m' \circ (l' + 2^{\nu-1}) = m' \circ l', \tag{1.115}$$

and then equation $(l' + 2^{\nu-1}) \circ m' = l' \circ m'$ follows directly. But Eq. (1.115) is just equation $l' \circ (m' + 2^{\nu-1}) = l' \circ m'$ where we have interchanged l and m (i.e. we have named m instead of l, and l instead of m), hence Eq. (1.115) is true always.

Note 1.3 That always $l \circ m = m \circ l$ can be proved by induction (of course, as superposition of strings, intuitively is clear): for $k = 0$ it holds; if it holds for k then, from (1.29), it holds evidently for $k + 1$ as well; so it holds for any k! □

8 Thus we have proved Eq.(1.87a) completely. Next we prove equation

$$(l' + 2^{\nu-1}) \circ (m' + 2^{\nu-1}) = l' \circ m' + 2^{\nu-1}, \tag{1.116}$$

which is (1.87b). We write it with ν' instead of ν as follows:

$$(l' + 2^{\nu'-1}) \circ (m' + 2^{\nu'-1}) = l' \circ m' + 2^{\nu'-1}. \tag{1.117}$$

We work as in Sec. 1.4**,4 with $l'+2^{\nu'-1}$ instead of l'. So, similarly, we find that instead of (1.93) we have now to prove the equation

$$(\alpha_{\mu(l'+2^{\nu'-1})\mu(m'+2^{\nu'-1})} - 1) \cdot 2^{2^k} + \alpha_{\lambda(l'+2^{\nu'-1})\lambda(m'+2^{\nu'-1})}$$
$$= (\alpha_{\mu(l')\mu(m')} - 1) \cdot 2^{2^k} + \alpha_{\lambda(l')\lambda(m')} + 2^{\nu'-1}. \tag{1.118}$$

The second part of (1.118) is written

$$(\alpha_{\mu(l')\mu(m')} - 1) \cdot 2^{2^k} + \alpha_{\lambda(l')\lambda(m')} + 2^{2^k+\nu-1}, \tag{1.119}$$

which is equal to

$$(\alpha_{\mu(l')\mu(m')} + 2^{\nu-1} - 1) \cdot 2^{2^k} + \alpha_{\lambda(l')\lambda(m')}. \tag{1.120}$$

So Eq. (1.118) becomes

$$(\alpha_{\mu(l'+2^{\nu'-1})\mu(m'+2^{\nu'-1})} - 1) \cdot 2^{2^k} + \alpha_{\lambda(l'+2^{\nu'-1})\lambda(m'+2^{\nu'-1})}$$
$$= (\alpha_{\mu(l')\mu(m')} + 2^{\nu-1} - 1) \cdot 2^{2^k} + \alpha_{\lambda(l')\lambda(m')}, \tag{1.121}$$

which is equivalent to the couple of equations

$$\alpha_{\mu(l'+2^{\nu'-1})\mu(m'+2^{\nu'-1})} = \alpha_{\mu(l')\mu(m')} + 2^{\nu-1}, \tag{1.122a}$$

and

$$\alpha_{\lambda(l'+2^{\nu'-1})\lambda(m'+2^{\nu'-1})} = \alpha_{\lambda(l')\lambda(m')}. \tag{1.122b}$$

[Eqs. (1.122) are the corresponding to Eqs. (1.94) of Sec. 1.4**,4 when we considered l' instead of $l' + 2^{\nu'-1}$ that we consider now]. Exactly as in Sec. 1.4**,5 we proved (1.95), here we prove its corresponding for l' and m' (using ν' instead of ν) i.e. we prove

$$\lambda(l' + 2^{\nu'-1}) = \lambda(l'), \tag{1.123a}$$

$$\lambda(m' + 2^{\nu'-1}) = \lambda(m'). \tag{1.123b}$$

Since we have proved (1.123) we have also proved (1.122b). Let us prove now (1.122a).

[The validity of (1.122a) is intuitively evident through Sec. 1.2,7 given the validity of (1.85b)].

Eq. (1.122a) equivalently is written

$$[\,\mu(l'+2^{\nu'-1})\,]\circ[\,\mu(m'+2^{\nu'-1})\,]=[\,\mu(l')\,]\circ[\,\mu(m')\,]+2^{\nu-1}. \quad (1.124)$$

Exactly as in Sec. 1.4**,6 we obtain now:

$$\mu(l')=l, \qquad \mu(m')=m, \quad (1.125a)$$

$$\mu(l'+2^{\nu'-1})=l+2^{\nu-1}, \qquad \mu(m'+2^{\nu'-1})=m+2^{\nu-1}. \quad (1.125b)$$

So, with Eqs. (1.125), equation (1.124) becomes

$$(l+2^{\nu-1})\circ(m+2^{\nu-1})=l\circ m+2^{\nu-1}. \quad (1.126)$$

Eq. (1.126) is true as we know from Eq. (1.85b)!

Thus we have proved (1.87b) completely.

9 Now all equations (1.87) have been proved. This means [see Sec. 1.3*,8] that if the property (1.85) holds for matrix k [(1.85) are written exactly for matrix k] then this property holds also for matrix $k+1$ [(1.87) is exactly property (1.85) written for matrix $k+1$]. Since property (1.85) holds also for the matrix with $k=0$ we deduce by induction that this property holds for the matrix of any k!

The content of the present section, i.e. of Sec. 1.4**, could be indeed a proof of (1.85) for any matrix k. But in any case the approach, described above, is interesting for its own sake.

It would be welcome to present the content of Sec. 1.1 and Sec. 1.2 in a "strict mathematical manner" perhaps exhibiting a clear *algebraic structure*. But in the following this content of the two sections will be used as it appears here.

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Chapter 2

Writing analytically the lines $2P_j^{\,i}$ with the help of the new style.

2.1 Some examples for presenting the superposition of the lines $2P_j^{\,i}$ with the help of the new style.

1 Now we are going to see in some examples how we use the new style, adopted in Chapter 1, for writing the superposition of the lines $2P_j^{\,i}$ in a better way facilitating things. We use the familiar, from Chapters 5,6, and 7 of the *Mathematical Diary 1993–1998*, notation and there is no need to repeat its meaning here.

As we have seen in Eqs. (6)–(8) as well as in Remark 1 of §1 of Chapter 7 of the *Mathematical Diary 1993–1998* (we just say *MD 1993–1998*) by $[+2P_j^{\,i}]$ we denote the superposition (or the *block* of the superposition) of the lines $2P_1^{\,i}, 2P_2^{\,i}, \cdots, 2P_j^{\,i}$. Equivalently here instead of $[+2P_j^{\,i}]$ we may write $|f(i|1,2,\cdots,j)|$ which is a form that reminds function.

We wish to construct $|f(1|1,2,\cdots,j)|$ with the help of the "new style" (the style described in Chapter 1). This will be not done in the present chapter. In this section we are going to see, as examples, the cases for $j=1$ and $j=2$, that is, $|f(1|1)|$ and $|f(1|1,2)|$.

2 Let us construct $|f(1|1)|$, which is $[+2P_1^{\,1}]$ or just the line $2P_1^{\,1}$, with the help of the "new style". This means, let us write $|f(1|1)|$ in the "new style".

We know [see Eq. (17) of Chapter 7 of *MD 1993–1998* and in other places] that

$$|f(1|1)| = 01111110, \tag{2.1}$$

with "period" $2^{\,3}$ which here is the "length" of the string i.e. the total number of the elementary digits 0 or 1 [see Remark 1 of Chapter 6 of *MD 1993–1998*]. Equivalently we write

$$|f(1|1)| = |01|11|11|10|^{2^3}, \tag{2.2}$$

with the "period", or length of the string, 2^3 written as upper index on the right. The string in (2.1) and (2.2) is made of digits 0 and 1 which may be considered as elementary strings of length one each ($1 = 2^{0}$). According to Section 1.1 and especially to (1.1) we may, instead of digits 0 and 1, write 1 and 2 respectively which are considered strings of one digit (strings of 1 or 2^{0} digits). Thus according to the conventions of Section 1.1 [see also (1.19)] we are in the case with $k = 0$, that is, the elementary strings 0 and 1, which are renamed 1 and 2 respectively, have length of one ($2^{k} = 2^{0} = 1$) digit. Thus instead of (2.2) we may write

$$|f(1|1)| = |12|22|22|21|^{2^3}, \quad (\text{case } \ k = 0), \tag{2.3}$$

(for case $k = 0$, that is, digits-strings made each of $2^k = 2^0 = 1$ elementary digit).

Briefly we may write (2.3) as

$$|f(1|1)| = (k = 0) \ \ | \overbrace{\ 1\ }^{2^0} 2|22|22|21|^{2^3}. \tag{2.4}$$

3 Renaming the pairs of digits in the string (2.3) or (2.4) according to (1.5) we obtain

$$|f(1|1)| = |2|4|4|3|^{2^3}, \quad (\text{case } \ k = 1), \tag{2.5}$$

and indeed we are in case $k = 1$ now since the length of each of the digits-strings 2 or 3 or 4 is 2 (that is 2^k for $k = 1$) elementary digits 1 or 2 [or better elementary digits 0 or 1 since we could equivalently apply the renaming (1.4) to (2.2) and obtain again (2.5)]. In accordance with (2.4) we can write (2.5) as

$$|f(1|1)| = (k = 1) \ \ | \overbrace{\ 2\ }^{2^1} |4|4|3|^{2^3}, \tag{2.6}$$

and also as

$$|f(1|1)| = (k = 1) \ \ | \overbrace{\ 2\ }^{2^1} 4|43|^{2^3}. \tag{2.7}$$

Renaming again the pairs of digits in the strings of (2.6) or (2.7) according to (1.10) we obtain similarly as before

$$|f(1|1)| = (k = 2) \ \ | \overbrace{\ 8\ }^{2^2} |\underline{15}|^{2^3}, \tag{2.8}$$

or equivalently

$$|f(1|1)| = (k=2) \;\; |\overbrace{8}^{2^2}\;\underline{15}|^{2^3}, \tag{2.9}$$

where we have underlined 15 to emphasize that it is the integer fifteen and not the string (concatenation) of digit 1 and digit 5! Here we are in case $k = 2$ since each of the units 8 and 15 has length (contains) 4, i.e. 2^k for $k = 2$, elementary digits 0 or 1 as we can analytically see in (1.11). This number of elementary digits again is emphasized in (2.9) by overbracing unit 8 as we have done also in (2.4) and in (2.6)–(2.8).

Renaming again the pair of units 8 and 15 in (2.9) by the analogous to (1.5) and (1.10) rule [in rule (1.5) strings $11, 12, \cdots, 22$ are renamed $1, 2, \cdots, 4$ respectively, in rule (1.10) strings $11, 12, \cdots, 44$ are renamed $1, 2, \cdots, 16$ respectively, so now analogously strings $11, 12, \cdots, (16)(16)$ are renamed strings $1, 2, \cdots, 256$ respectively] this pair 8 $\underline{15}$ is written $\underline{127}$, thus (2.9) becomes

$$|f(1|1)| = (k=3) \;\; |\overbrace{\underline{127}}^{2^3}|^{2^3}. \tag{2.10}$$

Again in (2.10) it is emphasized that we are in case $k = 3$, that is, unit 127 has length (contains) 8, which is 2^k for $k = 3$, elementary digits 0 or 1. This conventional form for presenting the strings will be followed hereafter without further explanations.

Note 2.1 Since this manner for successively "compressing" the strings [in (2.2)–(2.4) eight units, in (2.5)–(2.7) four units, in (2.8) and (2.9) two units, and finally in (2.10) one unit] will be met often in the following, we notice that for renaming the pairs of units there is no need to look at (1.5) or (1.10) or to construct the analogous correspondence for the other cases k. It is enough to use (1.25). Indeed, having a pair $\mu\lambda$ in case k its new name for case $k+1$ is just $(\mu - 1) \cdot 2^{2^k} + \lambda$. Let us see it in the example of (2.1)–(2.10).

For case $k = 0$, see (2.4), the pairs $\mu\lambda$ are 12,22,21 thus we have $(\mu, \lambda) = (1, 2), (2, 2), (2, 1)$ respectively and $(\mu - 1) \cdot 2^{2^k} + \lambda = 2, 4, 3$ respectively; so the new names for case $k = 1$ are 2,4,3 respectively, see (2.6).

For case $k = 1$, see (2.7), the pairs $\mu\lambda$ are 24 and 43 thus we have

$(\mu,\lambda)=(2,4),(4,3)$ respectively and $(\mu-1)\cdot 2^{2^k}+\lambda = 8, 15$ respectively; so the new names for case $k=2$ are 8 and 15 respectively, see (2.8).

For case $k=2$, see (2.9), the pair $\mu\lambda$ is 8 $\underline{15}$ thus we have $(\mu,\lambda)=(8,15)$ and $(\mu-1)\cdot 2^{2^k}+\lambda = 127$; so the new name for case $k=3$ is 127, see (2.10). $\square$

4 Now if we wish to construct $|f(1|1,2)|$, which is $[+2P^{\,1}_{\,2}]$, we just take line $2P^{\,1}_{\,1}$, which is denoted $|f(1|1)|$, four times and superpose it to line $2P^{\,1}_{\,2}$, which is denoted $|f(1|2)|$. We have seen analytically this superposition in (20) of Chapter 7 of the *MD 1993–1998*. Here this superposition may be performed with the help of the "new style" of Chapter 1. But before this some preliminary work is needed concerning the lines $2P^{\,i}_{\,j}$ or $|f(i|j)|$. For the moment let us see the line $2P^{\,1}_{\,2}$ or $|f(1|2)|$ as we have done for the line $2P^{\,1}_{\,1}$ or $|f(1|1)|$ previously.

We know [from Eq. (18) of Chapter 7 of the *MD 1993–1998* and from other places] that

$$|f(1|2)| = |0101|1111|0101|1111|1111|1010|1111|1010|\,, \tag{2.11}$$

which may be written simply as

$$|f(1|2)| = a_2c_2a_2c_2c_2b_2c_2b_2\,, \tag{2.12}$$

with $a_2=0101$, and $b_2=1010$, and $c_2=1111$.

Writing in (2.11) 1 and 2 instead of 0 and 1 and using the familiar style of presentation we obtain

$$|f(1|2)| = (k=0)\ \ |\overbrace{\ 1\ }^{2^0}212|2222|1212|2222|2222|2121|2222|2121|^{2^5}\,. \tag{2.13}$$

Renaming successively the pairs of units, as we have done in (2.4)–(2.10), according to (1.5) and (1.10) and generally according to (1.25) [see Note 2.1] we obtain

$$|f(1|2)| = (k=1)\ \ |\overbrace{\ 2\ }^{2^1}2|44|22|44|44|33|44|33|^{2^5}$$

$$= (k=1)\ \ |\overbrace{\ 2\ }^{2^1}244|2244|4433|4433|^{2^5}$$

$$= (k=2)\ \ |\overbrace{\ 6\ }^{2^2}, 16|6, 16|16, 11|16, 11|^{2^5}$$

$$= (k=2)\ \ |\overbrace{\ 6\ }^{2^2}, 16, 6, 16|16, 11, 16, 11|^{2^5}$$

$$= (k=3)\ \ |\overbrace{\ 96\ }^{2^3}, 96, 251, 251|^{2^5}$$

$$= (k=4)\ \ |\overbrace{24416}^{2^4}, 64251|^{2^5}$$

$$= (k=5)\ \ |\overbrace{1600125691}^{2^5}|^{2^5}\,. \qquad (2.14)$$

The descriptions and explanations for (2.4)–(2.10) hold also for (2.14). But now instead of underlining units to emphasize that they represent integers and not concatenation of digits we prefer, for simplicity, to separate the units by commas. For example in (2.14) instead of writing $|6\ \underline{16}|$, to denote that we have integers six and sixteen, we write $|6, 16|$ and instead of writing $|\underline{96}\ \underline{96}\ \underline{251}\ \underline{251}|$, to denote that we have integer 96 twice and integer 251 twice, we write $|96, 96, 251, 251|$. As for renaming the pairs of units thus writing them as new units we can see in (2.14) that, e.g. pair $|96, 96|$ in case $k = 3$ by (1.25) is written in case $k = 4$ as unit (integer) 24416: in (1.25) for $k = 3$ it is $\mu\lambda = |96, 96|$ which means $(\mu, \lambda) = (96, 96)$; so the new name of the pair in case $k = 4$, by (1.25), is $(\mu-1)\cdot 2^{2^k}+\lambda$ which is $(96-1)\cdot 2^{2^3} + 96$ which is 24416.

2.2 Some further examples. Emphasizing the binary nature of some things.

[We have here a parenthesis with some additional examples. This section could be part of Chapter 1 and, in fact, its content is not essential for what follows.]

1 Let us see again the superposition $l \circ m$ having in mind Section 1.1 and (1.25)–(1.29) [also Sec. 1.3*,7 and Fig. 1.2 might be useful].

For the general l and m, which are in case k [this means that l and m have length of totally 2^k elementary digits 0 or 1 each, and that they are chosen among totally 2^{2^k} species], we obtain from (1.29)

$$l \circ m = [\,\mu(l) \circ \mu(m) - 1\,] \cdot 2^{2^{k-1}} + \lambda(l) \circ \lambda(m)\,, \qquad (2.15)$$

where instead of writing μ_1 and λ_1 we have written $\mu(l)$ and $\lambda(l)$ respectively, and instead of μ_2 and λ_2 we have $\mu(m)$ and $\lambda(m)$ respectively (we have seen this notation also in Sec. 1.3*,6 as well as during the whole Sec. 1.4**).

Now writing (2.15) for $\mu(l) \circ \mu(m)$ and $\lambda(l) \circ \lambda(m)$, instead of $l \circ m$, we obtain respectively

$$\mu(l) \circ \mu(m) = [\,\mu(\mu(l)) \circ \mu(\mu(m)) - 1\,] \cdot 2^{2^{k-2}} + \lambda(\mu(l)) \circ \lambda(\mu(m))\,, \qquad (2.16a)$$

and

$$\lambda(l) \circ \lambda(m) = [\,\mu(\lambda(l)) \circ \mu(\lambda(m)) - 1\,] \cdot 2^{2^{k-2}} + \lambda(\lambda(l)) \circ \lambda(\lambda(m))\,. \qquad (2.16b)$$

Similarly if we write (2.15) for $\mu(\mu(l)) \circ \mu(\mu(m))$ and $\lambda(\mu(l)) \circ \lambda(\mu(m))$ and $\mu(\lambda(l)) \circ \mu(\lambda(m))$ and $\lambda(\lambda(l)) \circ \lambda(\lambda(m))$, instead of $l \circ m$, we obtain respectively

$$\mu(\mu(l)) \circ \mu(\mu(m)) = [\,\mu(\mu(\mu(l))) \circ \mu(\mu(\mu(m))) - 1\,] \cdot 2^{2^{k-3}} + \lambda(\mu(\mu(l))) \circ \lambda(\mu(\mu(m)))\,, \qquad (2.17a)$$

$$\lambda(\mu(l)) \circ \lambda(\mu(m)) = [\,\mu(\lambda(\mu(l))) \circ \mu(\lambda(\mu(m))) - 1\,] \cdot 2^{2^{k-3}} + \lambda(\lambda(\mu(l))) \circ \lambda(\lambda(\mu(m)))\,, \qquad (2.17b)$$

$$\mu(\lambda(l))) \circ \mu(\lambda(m))) = [\,\mu(\mu(\lambda(l))) \circ \mu(\mu(\lambda(m))) - 1\,] \cdot 2^{2^{k-3}} + \lambda(\mu(\lambda(l))) \circ \lambda(\mu(\lambda(m)))\,, \qquad (2.17c)$$

$$\lambda(\lambda(l)) \circ \lambda(\lambda(m)) = [\,\mu(\lambda(\lambda(l))) \circ \mu(\lambda(\lambda(m))) - 1\,] \cdot 2^{2^{k-3}} + \lambda(\lambda(\lambda(l))) \circ \lambda(\lambda(\lambda(m)))\,. \qquad (2.17d)$$

We may continue the procedure in the same way.

If we want to avoid the parentheses we may write (2.15)–(2.17) as follows. Eq. (2.15) is written

$$l \circ m = (\mu l \circ \mu m - 1) \cdot 2^{2^{k-1}} + \lambda\, l \circ \lambda m\,, \qquad (2.18)$$

Eqs. (2.16) are written respectively

$$\mu l \circ \mu m = (\mu^2 l \circ \mu^2 m - 1) \cdot 2^{2^{k-2}} + \lambda\mu l \circ \lambda\mu m\,, \tag{2.19a}$$

$$\lambda\, l \circ \lambda m = (\mu\lambda\, l \circ \mu\lambda m - 1) \cdot 2^{2^{k-2}} + \lambda^2 l \circ \lambda^2 m\,, \tag{2.19b}$$

and Eqs. (2.17) are written respectively

$$\mu^2 l \circ \mu^2 m = (\mu^3 l \circ \mu^3 m - 1) \cdot 2^{2^{k-3}} + \lambda\mu^2 l \circ \lambda\mu^2 m\,, \tag{2.20a}$$

$$\lambda\mu l \circ \lambda\mu m = (\mu\lambda\mu l \circ \mu\lambda\mu m - 1) \cdot 2^{2^{k-3}} + \lambda^2\mu l \circ \lambda^2\mu m\,, \tag{2.20b}$$

$$\mu\lambda\, l \circ \mu\lambda m = (\mu^2\lambda\, l \circ \mu^2\lambda m - 1) \cdot 2^{2^{k-3}} + \lambda\mu\lambda\, l \circ \lambda\mu\lambda m\,, \tag{2.20c}$$

$$\lambda^2 l \circ \lambda^2 m = (\mu\lambda^2 l \circ \mu\lambda^2 m - 1) \cdot 2^{2^{k-3}} + \lambda^3 l \circ \lambda^3 m\,. \tag{2.20d}$$

2 Let us consider $7 \circ 12$ as example for $l \circ m$ in case $k = 2$ [see (1.10),(1.11), and (1.14)]. Strings 7 and 12 are chosen from the set of strings $\{1, 2, \cdots, 16\}$ and in (1.14) we see that $7 \circ 12 = 3$. We write the following set of equations which are described next:

$$\begin{array}{lrcll} (k=2) & 7 \circ 12 & = & (2 \circ 3 - 1) \cdot 4 + 3 \circ 4 & = \ 3 \\ (k=1) & 2 \circ 3 & = & (1 \circ 2 - 1) \cdot 2 + 2 \circ 1 & = \ 1 \\ (k=1) & 3 \circ 4 & = & (2 \circ 2 - 1) \cdot 2 + 1 \circ 2 & = \ 3 \\ (k=0) & 1 \circ 2 & = & 1 & = \ 1 \\ (k=0) & 2 \circ 1 & = & 1 & = \ 1 \\ (k=0) & 2 \circ 2 & = & 2 & = \ 2 \end{array} \tag{2.21}$$

In (2.21) the final results (e.g. $7 \circ 12 = 3$, $2 \circ 3 = 1$, and so on) can be taken directly from (1.14),(1.8), and (1.3). But the final results as well as the intermediary steps can also be deduced from (1.25)–(1.29) [see also the example (1.30)–(1.33)]. Thus for the first of equations (2.21) it is $k = 2$ and $l = 7$ and $m = 12$ hence from (1.27) [where we put $k = 1$] we obtain $(\mu_1, \lambda_1) = (2, 3)$ and $(\mu_2, \lambda_2) = (3, 4)$ hence from (1.29) we obtain $7 \circ 12 = (2 \circ 3 - 1) \cdot 2^{2^1} + 3 \circ 4 = (2 \circ 3 - 1) \cdot 4 + 3 \circ 4$ and this is the intermediary step in the first equation of (2.21). For computing $7 \circ 12$ from this step we must first compute $2 \circ 3$ and $3 \circ 4$. These are computed in the second and third equation of (2.21) [see the intermediary steps of these two equations which are constructed as the corresponding part of the first equation] but for a complete computation we need first to compute $1 \circ 2$, $2 \circ 1$, and $2 \circ 2$ which are elementary hence are obtained directly from (1.3) and are written in the fourth, fifth, and sixth of equations (2.21). So inserting the values $1 \circ 2 = 1$, $2 \circ 1 = 1$, and $2 \circ 2 = 2$ into the intermediary steps of the second and third of equations (2.21) we obtain the values $2 \circ 3 = 1$ and $3 \circ 4 = 3$ which, similarly, inserted into the intermediary step of the first of equations (2.21) give finally the value $7 \circ 12 = 3$.

We could insert $2 \circ 3$ and $3 \circ 4$, from the second and third of equations (2.21) [intermediary steps], into the first of these equations which might be written

$$(k=2)\ \ 7\circ 12 = [\,((1\circ 2-1)\cdot 2^{2^0}+2\circ 1)-1\,]\cdot 2^{2^1}+[\,(2\circ 2-1)\cdot 2^{2^0}+1\circ 2\,]. \tag{2.22}$$

In fact, in (2.22) we have used (1.29) twice successively.

3 Let us see an alternative computation of $7 \circ 12$ based on the structure of (1.14) [we might have in mind also Sec. 1.3*,7 and especially Fig. 1.2 as well as (1.84) and property (1.85)] and especially on the separation of the block (matrix) into four subblocks (submatrices) namely I,II,III,IV [as it is described analytically in Sec. 1.3* and mainly in (1.80)]. We consider the computation step-by-step on the matrix (1.14).

The strings 7 and 12 are in case $k = 2$ because they are composed, each, of $2^k = 2^2 = 4$ elementary digits 0 or 1. Then:

a) The site $7\circ 12$, which is in the submatrix II [submatrix $(1, 2, \cdots, 8) \times (9, 10, \cdots, 16)$ with dimension, i.e. length of side, 2^3] of matrix (1.14) with dimension 2^4 [and is equal to 3], because of the (theoretically investigated) structure of (1.14) is equal to site $7 \circ 4$ of the submatrix I [submatrix $(1, 2, \cdots, 8) \times (1, 2, \cdots, 8)$] of (1.14).

b) The site $7 \circ 4$ of the submatrix I (with dimension 2^3) may be considered as lying in the submatrix III [submatrix $(5, 6, 7, 8) \times (1, 2, 3, 4)$ with dimension 2^2] of the matrix $(1, 2, \cdots, 8)\times(1, 2, \cdots, 8)$ [which is the submatrix I of matrix (1.14)].

c) Because of the structure of (1.14) the site $7 \circ 4$ considered in the submatrix III [submatrix $(5, 6, 7, 8)\times(1, 2, 3, 4)$] of matrix $(1, 2, \cdots, 8) \times(1, 2, \cdots, 8)$ is equal to the site $3\circ 4$ of the submatrix I [submatrix $(1, 2, 3, 4) \times (1, 2, 3, 4)$ of dimension 2^2] of matrix $(1, 2, \cdots, 8) \times$

$(1, 2, \cdots, 8)$.

d) The site $3 \circ 4$ of submatrix $(1, 2, 3, 4) \times (1, 2, 3, 4)$ may be considered as lying in the submatrix IV [submatrix $(3, 4) \times (3, 4)$ of dimension 2^1] of matrix $(1, 2, 3, 4) \times (1, 2, 3, 4)$.

e) The site $3 \circ 4$ of the submatrix $(3, 4) \times (3, 4)$, because of the structure of (1.14), is equal to the site $1 \circ 2$ of submatrix I [submatrix $(1, 2) \times (1, 2)$] of the matrix $(1, 2, 3, 4) \times (1, 2, 3, 4)$ plus 2^1 (that is $3 \circ 4 = 1 \circ 2 + 2^1$).

f) The site $1 \circ 2$ of the submatrix $(1, 2) \times (1, 2)$ may be considered as the (trivial) submatrix II [submatrix 1×2 of dimension 2^0] of matrix $(1, 2) \times (1, 2)$.

g) Finally, the site (trivial submatrix) $1 \circ 2$, because of the structure of (1.14), is equal to the trivial submatrix (site) I [submatrix 1×1] of the matrix $(1, 2) \times (1, 2)$.

The above procedure may be compactly presented as follows:

$$\overbrace{7 \circ 12}^{II,2^3} = \overbrace{7 \circ 4}^{I,2^3} = \overbrace{7 \circ 4}^{III,2^2} = \overbrace{3 \circ 4}^{I,2^2} = \overbrace{3 \circ 4}^{IV,2^1} = \overbrace{1 \circ 2}^{I,2^1} + 2^1$$

$$= \overbrace{1 \circ 2}^{II,2^0} + 2^1 = \overbrace{1 \circ 1}^{I,2^0} + 2^1 = 1 + 2^1 = 3. \qquad (2.23)$$

All steps (a)–(g) appear in (2.23). For example in (2.23) overbracing $7 \circ 12$ by $II, 2^3$ means that we consider site $7 \circ 12$ as being in the submatrix II (of dimension 2^3) of matrix (1.14) [see step (a)]. Then this site is equal to site $7 \circ 4$, which is considered in submatrix I (of dimension 2^3) of matrix (1.14), and so on.

4 Next we write again steps (a)–(g), as well as (2.23), in an "array-form". First we write them in "decimal" style and second in "binary" style.

First the *decimal* style:

$$
\begin{array}{llll}
(k=2) & \begin{array}{l} 2^4, 2^3 \\ \end{array} & \begin{array}{c} 7 \circ 12 \\ 7 \circ 4 \end{array} & \left.\begin{array}{r} II \\ I \end{array}\right\} 2^3 \\
 & \begin{array}{l} 2^3, 2^2 \\ \end{array} & \begin{array}{c} 7 \circ 4 \\ 3 \circ 4 \end{array} & \left.\begin{array}{r} III \\ I \end{array}\right\} 2^2 \\
 & \begin{array}{l} 2^2, 2^1 \\ \end{array} & \begin{array}{c} 3 \circ 4 \\ 1 \circ 2 \end{array} & \left.\begin{array}{r} IV \\ I \end{array}\right\} \underline{2^1} \\
 & \begin{array}{l} 2^1, 2^0 \\ \end{array} & \begin{array}{c} 1 \circ 2 \\ \underline{1 \circ 1} \end{array} & \left.\begin{array}{r} II \\ I \end{array}\right\} 2^0
\end{array}
\tag{2.24}
$$

thus

$$
7 \circ 12 = 1 + 2^1 = 3. \tag{2.25}
$$

In (2.24) in front of the superpositions we have written the dimensions of the matrices and submatrices. For example, in front of superposition $7 \circ 12$ there are the dimensions 2^3 and 2^4 of the submatrix and matrix respectively to which we refer. In front of the next superposition $7 \circ 4$ (in I) is written nothing and this means that we have again the previous dimensions 2^3 and 2^4. The same things hold similarly for the other superpositions.

Next we write again (2.24) but with the integers in the superpositions expressed in the *binary* system. For simplicity we omit the dimensions in front of the superpositions.

$$
\begin{array}{rcll}
\begin{array}{r} 1\,1\,1 \\ 1\,1\,1 \end{array} & \begin{array}{c} \circ \\ \circ \end{array} & \begin{array}{l} 1\,1\,0\,0 \\ 1\,0\,0 \end{array} & \left.\begin{array}{r} II \\ I \end{array}\right\} 2^3 \\
\begin{array}{r} 1\,1\,1 \\ 1\,1 \end{array} & \begin{array}{c} \circ \\ \circ \end{array} & \begin{array}{l} 1\,0\,0 \\ 1\,0\,0 \end{array} & \left.\begin{array}{r} III \\ I \end{array}\right\} 2^2 \\
\begin{array}{r} 1\,1 \\ 1 \end{array} & \begin{array}{c} \circ \\ \circ \end{array} & \begin{array}{l} 1\,0\,0 \\ 1\,0 \end{array} & \left.\begin{array}{r} IV \\ I \end{array}\right\} \underline{2^1} \\
\begin{array}{r} 1 \\ \underline{1} \end{array} & \begin{array}{c} \circ \\ \circ \end{array} & \begin{array}{l} 1\,0 \\ \underline{1} \end{array} & \left.\begin{array}{r} II \\ I \end{array}\right\} 2^0
\end{array}
\tag{2.26}
$$

[for example when in the first row of (2.26) we write $111 \circ 1100$ we mean, as it is known from the binary representation of the integers, $(1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0) \circ (1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0)$ which is $7 \circ 12$, and when in the fourth row we write $11 \circ 100$ we mean $(1 \cdot 2^1 + 1 \cdot 2^0) \circ (1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0)$ which is $3 \circ 4$]

thus

$$
7 \circ 12 = 1 + 2^1 = 3. \tag{2.27}
$$

The binary style (2.26) has indeed something expressive. Nevertheless, for our work we rather prefer the decimal style (2.24)!

5 So far, in relations and applications we have accepted for l that

$$l \in \{1, 2, \cdots, 2^{2^k}\}, \tag{2.28}$$

see for example (1.20). But alternatively we may accept that

$$l \in \{0, 1, 2, \cdots, 2^{2^k} - 1\}. \tag{2.29}$$

In such case instead of e.g. $7 \circ 12 = 3$ we write $6 \circ 11 = 2$.

If we accept for l and m that

$$l, m \in \{0, 1, 2, \cdots, 2^{2^k} - 1\} \tag{2.30}$$

instead of (1.20), then we can write

$$l' = \mu \cdot 2^{2^k} + \lambda \tag{2.31}$$

instead of the first of equations (1.27) [instead of μ_1 and λ_1 here we write just μ and λ respectively, or $\mu(l')$ and $\lambda(l')$ respectively] with μ and λ taking values from the set in (2.30) [as l and m] and being analytically

$$\mu = quot(l'/2^{2^k}) \quad \text{and} \quad \lambda = rem(l'/2^{2^k}). \tag{2.32}$$

Now instead of (1.29) we have

$$\begin{aligned} l' \circ m' &= [\,\mu(l') \cdot 2^{2^k} + \lambda(l')\,] \circ [\,\mu(m') \cdot 2^{2^k} + \lambda(m')\,] \\ &= [\,\mu(l') \circ \mu(m')\,] \cdot 2^{2^k} + [\,\lambda(l') \circ \lambda(m')\,]. \end{aligned} \tag{2.33}$$

In the following parts of the present chapter the convention (2.30), for the values of l and m, will be often used because the so-resulting formulae are simpler.

6 Analysing $7 \circ 12$ we have obtained (2.24)–(2.27). Doing the same thing but now with the convention (2.30), for the values of l and m, instead of (1.20) we analyse $6 \circ 11$ (instead of $7 \circ 12$) and instead of (2.24)–(2.27) we obtain:

in the *decimal* style

$$
\begin{array}{llll}
(k=2) & 2^4, 2^3 & 6 \circ 11 & II \\
 & & 6 \circ 3 & I \\
 & 2^3, 2^2 & 6 \circ 3 & III \\
 & & 2 \circ 3 & I \\
 & 2^2, 2^1 & 2 \circ 3 & IV \\
 & & 0 \circ 1 & I \\
 & 2^1, 2^0 & 0 \circ 1 & II \\
 & & \underline{0 \circ 0} & I
\end{array}
\left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\right.
\begin{array}{l}
\left.\begin{array}{l} \\ \\ \end{array}\right\} 2^3 \\
\left.\begin{array}{l} \\ \\ \end{array}\right\} 2^2 \\
\left.\begin{array}{l} \\ \\ \end{array}\right\} \underline{2^1} \\
\left.\begin{array}{l} \\ \\ \end{array}\right\} 2^0
\end{array}
\tag{2.34}
$$

thus

$$6 \circ 11 = 0 + 2^1 = 2\,; \tag{2.35}$$

and in the *binary* style

$$
\begin{array}{rcll}
1\,1\,0 & \circ & 1\,0\,1\,1 & II \\
1\,1\,0 & \circ & 1\,1 & I \\
1\,1\,0 & \circ & 1\,1 & III \\
1\,0 & \circ & 1\,1 & I \\
1\,0 & \circ & 1\,1 & IV \\
0 & \circ & 1 & I \\
0 & \circ & 1 & II \\
\underline{0} & \circ & \underline{0} & I
\end{array}
\begin{array}{l}
\left.\begin{array}{l} \\ \\ \end{array}\right\} 2^3 \\
\left.\begin{array}{l} \\ \\ \end{array}\right\} 2^2 \\
\left.\begin{array}{l} \\ \\ \end{array}\right\} \underline{2^1} \\
\left.\begin{array}{l} \\ \\ \end{array}\right\} 2^0
\end{array}
\tag{2.36}
$$

thus

$$6 \circ 11 = 0 + 2^1 = 2\,. \tag{2.37}$$

The convention (1.20) is more suitable, from some aspects, for representing the strings and their superpositions. But when we consider the binary representation of the integers the convention (2.30) is better.

2.3 Writing down analytically the lines $2P\,^1_1, 2P\,^2_1, 2P\,^3_1, \cdots$ and $2P\,^1_2, 2P\,^2_2, 2P\,^3_2, \cdots$ and $2P\,^1_3, 2P\,^2_3, 2P\,^3_3, \cdots$ and so on, or equivalently the lines $|f(1|1)|, |f(2|1)|, |f(3|1)|, \cdots$ and $|f(1|2)|, |f(2|2)|, |f(3|2)|, \cdots$ and $|f(1|3)|, |f(2|3)|, |f(3|3)|, \cdots$ and so on respectively, with the help of the "new style" described in Chapter 1.

1 The essential part of the present Chapter 2 starts here. The notation as well as the style of presenting things is familiar from Section 2.1 so we proceed directly without further explanations.

Wishing to produce the superpositions of the lines $2P^{\,i}_{\,j}$ in the style of Chapter 1 it is good first to write each such line separately in this style. Let us present some such lines which are taken from Chapters 5,6, and 7 of the *Mathematical Diary 1993–1998* [these elementary lines appear all together in Figure 2 of Chapter 5 of the *MD 1993–1998*].

$$|f(1|1)| = 2P^{\,1}_{\,1} = |01|11|11|10|^{2^3} \tag{2.38a}$$

$$|f(2|1)| = 2P^{\,2}_{\,1} = |0011|1111|1111|1100|^{2^4} \tag{2.38b}$$

$$|f(3|1)| = 2P^{\,3}_{\,1} = |00001111|11111111|11111111|11110000|^{2^5} \tag{2.38c}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$|f(1|2)| = 2P^{\,1}_{\,2} = |0101|1111|0101|1111|1111|1010|1111|1010|^{2^5} \tag{2.39a}$$

$$\begin{aligned} |f(2|2)| = 2P^{\,2}_{\,2} = \ & |00110011|11111111|00110011|11111111| \\ & |11111111|11001100|11111111|11001100|^{2^6} \end{aligned} \tag{2.39b}$$

$$\begin{aligned} |f(3|2)| = 2P^{\,3}_{\,2} = \ & |0000111100001111|1111111111111111| \\ & |0000111100001111|1111111111111111| \\ & |1111111111111111|1111000011110000| \\ & |1111111111111111|1111000011110000|^{2^7} \end{aligned} \tag{2.39c}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\begin{aligned} |f(1|3)| = 2P^{\,1}_{\,3} = \ & |01010101|11111111|01010101|11111111| \\ & |01010101|11111111|01010101|11111111| \\ & |11111111|10101010|11111111|10101010| \\ & |11111111|10101010|11111111|10101010|^{2^7} \end{aligned} \tag{2.40a}$$

$$
\begin{aligned}
|f(2|3)| = 2P\,^{2}_{3} = \;& |0011001100110011|1111111111111111| \\
& |0011001100110011|1111111111111111| \\
& |0011001100110011|1111111111111111| \\
& |0011001100110011|1111111111111111| \\
& |1111111111111111|1100110011001100| \\
& |1111111111111111|1100110011001100| \\
& |1111111111111111|1100110011001100| \\
& |1111111111111111|1100110011001100|^{2^8}
\end{aligned}
\tag{2.40b}
$$

$$
\begin{aligned}
|f(3|3)| = 2P\,^{3}_{3} = \;& |00001111000011110000111100001111| \\
& |11111111111111111111111111111111| * * \bar{*}\,^{2^9}
\end{aligned}
\tag{2.40c}
$$

$$
\vdots \qquad\qquad \vdots \qquad\qquad \vdots
$$

$$
\cdot \qquad\qquad \cdot \qquad\qquad \cdot
$$

Note 2.2 In (2.40c) at the end of the string there are the symbols $*$ and $\bar{*}$. The meaning of these symbols is as described in Note 5 and Comment 9 of Chapter 7 of the *MD 1993–1998*. Practically here this means that: a string with symbol $*$ at its end is the concatenation of this string with a replica of itself; a string with symbol $\bar{*}$ at its end is the concatenation of this string with the same string written in reverse order. For example, by 0101$*$ we mean the concatenation of 0101 with 0101 which is 01010101, and by 0101$\bar{*}$ we mean the concatenation of 0101 with 1010 which is 01011010. □

2 In (2.38)–(2.40) we can, according to Section 1.1, replace digits 0 and 1 respectively by digits 1 and 2 (we are in case $k = 0$). But since the form of the strings is clear there is no need to do it here analytically.

Now renaming successively the pairs of units in the strings of (2.38)–(2.40), as we have seen in Section 2.1, we can transform the appearance of (2.38)–(2.40). We do it analytically in the following and since the procedure is adequately described in Sec. 2.1 there is no need for additional descriptions and explanations. So we present the transformed (2.38)–(2.40) directly.

Next we write the transformed (2.38)–(2.40) for case $k = 1$. The transformation is based on (1.4).

$$|f(1|1)| = |24|43|^{2^3} \tag{2.41a}$$

$$|f(2|1)| = |14|44|44|41|^{2^4} \tag{2.41b}$$

$$|f(3|1)| = |1144|4444|4444|4411|^{2^5} \tag{2.41c}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$|f(1|2)| = |2244|2244|4433|4433|^{2^5} \tag{2.42a}$$

$$|f(2|2)| = |1414|4444|1414|4444|4444|4141|4444|4141|^{2^6} \tag{2.42b}$$

$$\begin{aligned} |f(3|2)| = \;& |11441144|44444444|11441144|44444444| \\ & |44444444|44114411|44444444|44114411|^{2^7} \end{aligned} \tag{2.42c}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\begin{aligned} |f(1|3)| = \;& |22224444|22224444|22224444|22224444| \\ & |44443333|44443333|44443333|44443333|^{2^7} \end{aligned} \tag{2.43a}$$

$$|f(2|3)| = |14141414|44444444| \ast\ast\bar{\ast}^{\,2^8} \tag{2.43b}$$

$$|f(3|3)| = |1144114411441144|4444444444444444| \ast\ast\bar{\ast}^{\,2^9} \tag{2.43c}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\cdot \qquad\qquad \cdot \qquad\qquad \cdot$$

3 Next we write the transformed (2.41)–(2.43) for the case $k = 2$. The transformation is based on (1.10).

$$|f(1|1)| = |8, 15|^{2^3} \tag{2.44a}$$

$$|f(2|1)| = |4, 16, 16, 13|^{2^4} \tag{2.44b}$$

$$|f(3|1)| = |1, 16, 16, 16|16, 16, 16, 1|^{2^5} \tag{2.44c}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$|f(1|2)| = |6, 16, 6, 16|16, 11, 16, 11|^{2^5} \tag{2.45a}$$

$$|f(2|2)| = |4, 4, 16, 16|4, 4, 16, 16|16, 16, 13, 13|16, 16, 13, 13|^{2^6} \tag{2.45b}$$

$$|f(3|2)| = \begin{array}{l} |1,16,1,16,16,16,16,16,1,16,1,16,16,16,16,16| \\ |16,16,16,16,16,1,16,1,16,16,16,16,16,1,16,1|^{2^7} \end{array} \tag{2.45c}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$|f(1|3)| = \begin{array}{l} |6,6,16,16,6,6,16,16,6,6,16,16,6,6,16,16| \\ |16,16,11,11,16,16,11,11,16,16,11,11,16,16,11,11|^{2^7} \end{array} \tag{2.46a}$$

$$|f(2|3)| = (|4,4,4,4,16,16,16,16|{*}{*})(|16,16,16,16,13,13,13,13|{*}{*})^{2^8} \tag{2.46b}$$

$$|f(3|3)| = |1,16,1,16,1,16,1,16,16,16,16,16,16,16,16,16| \ast\ast\bar{\ast}^{\,2^9} \tag{2.46c}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$. \qquad . \qquad .$$

4 Next we write the transformed (2.44)–(2.46) for the case $k = 3$. For the transformation it is good to have in mind Note 2.1.

$$|f(1|1)| = |127|^{2^3} \tag{2.47a}$$

$$|f(2|1)| = |64,253|^{2^4} \tag{2.47b}$$

$$|f(3|1)| = |16,256,256,241|^{2^5} \tag{2.47c}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$|f(1|2)| = |96,96,251,251|^{2^5} \tag{2.48a}$$

$$|f(2|2)| = |52,256,52,256,256,205,256,205|^{2^6} \tag{2.48b}$$

$$|f(3|2)| = \begin{array}{l} |16,16,256,256,16,16,256,256| \\ |256,256,241,241,256,256,241,241|^{2^7} \end{array} \tag{2.48c}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$|f(1|3)| = \begin{array}{l} |86,256,86,256,86,256,86,256| \\ |256,171,256,171,256,171,256,171|^{2^7} \end{array} \tag{2.49a}$$

$$|f(2|3)| = (|52,52,256,256| \ast\ast)(|256,256,205,205| \ast\ast)^{2^8} \tag{2.49b}$$

$$|f(3|3)| = \begin{array}{l}(|16,16,16,16,256,256,256,256| **) \\ (|256,256,256,256,241,241,241,241| **)^{2^9}\end{array} \qquad (2.49c)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$. \qquad . \qquad .$$

2.4 Determining by recursive algorithm the basic units of which the lines $|f(1|j)|$ [i.e. the lines $2P^{\,1}_{\,j}$] are composed.

1 As we can see in Sec. 2.3 in each $|f(i|j)|$ there are at most three different units [for example, in $|f(1|3)|$ for $k = 3$, see (2.49a), there are units 86,256, and 171]. We seek the structure of these units.

We start by investigating the lines $|f(1|j)|$, that is, the lines $2P^{\,1}_{\,j}$. This is performed with the help of the examples of Sec. 2.3. We write analytically the units, existing in the various $|f(1|j)|$'s, for the cases $k = 0, 1, 2, \cdots$. These units are presented in their "detailed" form as in (2.38)–(2.40) [with 1 and 2 instead of 0 and 1], and in their successively "compactified" form as in (2.41)–(2.49). For this "compactification" of the units, for the general k, we have in mind Note 2.1 (as well as Sec. 1.1 and Sec. 1.2).

Next we write down directly the lines $|f(1|1)|$, $|f(1|2)|$, and $|f(1|3)|$ as well as their corresponding units successively "compactified".

$$|f(1|1)| = |01111110|^{2^3} = |12222221|^{2^3} \qquad (2.50a)$$

with units

$$\begin{array}{lllll}
\text{for } k=0: & 1,2 & \equiv 1,2 & \in \{1,2\} & = \{1,2,\cdots,2^{2^0}\} \\
\text{for } k=1: & 12,22,21 & \equiv 2,4,3 & \in \{1,2,\cdots,4\} & = \{1,2,\cdots,2^{2^1}\} \\
\text{for } k=2: & 1222,2221 & \equiv 8,15 & \in \{1,2,\cdots,16\} & = \{1,2,\cdots,2^{2^2}\} \\
\text{for } k=3: & 12222221 & \equiv 127 & \in \{1,2,\cdots,256\} & = \{1,2,\cdots,2^{2^3}\}
\end{array} \qquad (2.50b)$$

$$\begin{aligned} |f(1|2)| &= |01011111010111111111101011111010| \\ &= |12122222121222222222212122222121|^{2^5} \end{aligned} \qquad (2.51a)$$

with units

$$\begin{array}{llll} \text{for } k=0: & 1,2 & \equiv 1,2 & \in \{1,2,\cdots,2^{2^0}\} \\ \text{for } k=1: & 12,22,21 & \equiv 2,4,3 & \in \{1,2,\cdots,2^{2^1}\} \\ \text{for } k=2: & 1212,2222,2121 & \equiv 6,16,11 & \in \{1,2,\cdots,2^{2^2}\} \\ \text{for } k=3: & 12122222,22222121 & \equiv 96,251 & \in \{1,2,\cdots,2^{2^3}\} \end{array}$$

$$\begin{array}{ll} \text{for } k=4: & 1212222212122222, 2222212122222121 \\ & \equiv 95\cdot 256+96, 250\cdot 256+251 \\ \text{for } k=5: & 12122222121222222222212122222121 \\ & \equiv (95\cdot 256+96-1)\cdot 2^{16}+(250\cdot 256+251) \end{array} \tag{2.51b}$$

$$\begin{aligned} |f(1|3)| = \ & |0101010111111111010101011111111| \\ & |0101010111111111010101011111111| \\ & |1111111110101010111111110101010| \\ & |1111111110101010111111110101010|^{2^7} \\ = \ & |1212121222222222121212122222222| \\ & |1212121222222222121212122222222| \\ & |2222222221212121222222221212121| \\ & |2222222221212121222222221212121|^{2^7} \end{aligned} \tag{2.52a}$$

with units

$$\begin{array}{llll} \text{for } k=0: & 1,2 & \equiv 1,2 & \in \{1,2,\cdots,2^{2^0}\} \\ \text{for } k=1: & 12,22,21 & \equiv 2,4,3 & \in \{1,2,\cdots,2^{2^1}\} \\ \text{for } k=2: & 1212,2222,2121 & \equiv 6,16,11 & \in \{1,2,\cdots,2^{2^2}\} \end{array}$$

$$\begin{array}{lll} \text{for } k=3: & 12121212, 22222222, 21212121 & \equiv 86,256,171 \\ & & \in \{1,2,\cdots,2^{2^3}\} \end{array}$$

$$\begin{aligned}
\text{for } k=4: \quad & 1212121222222222, 2222222221212121 \\
& \equiv 86 \cdot 256, 255 \cdot 256 + 171 \\
\text{for } k=5: \quad & 12121212222222221212121222222222, \\
& 22222222212121212222222221212121 \\
& \equiv (86 \cdot 256 - 1) \cdot 2^{16} + 86 \cdot 256, \\
& \quad (255 \cdot 256 + 171 - 1) \cdot 2^{16} + (255 \cdot 256 + 171) \\
\text{for } k=6: \quad & 12121212222222221212121222222222 \\
& 12121212222222221212121222222222, \\
& 22222222212121212222222221212121 \\
& 22222222212121212222222221212121 \equiv \cdots \\
\text{for } k=7: \quad & 12121212222222221212121222222222 \\
& 12121212222222221212121222222222 \\
& 22222222212121212222222221212121 \\
& 22222222212121212222222221212121 \equiv \cdots
\end{aligned} \tag{2.52b}$$

2 Let us see the units appearing in (2.50b),(2.51b), and (2.52b) in another manner.

Using as basic example the elementary line $|f(1|1)|$ we see from (2.50b) that:

for $k = 0$ there are units 1 and 2;

for $k = 1$ there are units 12,22, and 21 which are renamed $(1 - 1) \cdot 2 + 2, (2 - 1) \cdot 2 + 2$, and $(2 - 1) \cdot 2 + 1$ respectively which are equal to 2,4, and 3 respectively;

for $k = 2$ there are units 24 and 43 (or more elementarily 1222 and 2221) which are renamed $(2 - 1) \cdot 4 + 4$ and $(4 - 1) \cdot 4 + 3$ respectively which are equal to 8 and 15 respectively;

and so on.

Wishing to emphasize the way in which the units are formed we can denote these units by increasing integers instead of their names. For example the units in (2.50b) may be denoted: for $k = 0$ 1 and 2 instead of (trivial case) 1 and 2; for $k = 1$ 1,2, and 3 instead of 12,22, and 21 or 2,4, and 3; for $k = 2$ 1 and 2 instead of 1222 and 2221 or 8 and 15; for $k = 3$ 1 instead of 12222221 or 127. This schematically appears in the following array.

$$\begin{array}{ll}
\multicolumn{2}{c}{units\ for\ line\ |f(1|1)|:} \\
k=0 & 1,\ \ 2 \\
 & 1\ \ \ 2 \\
k=1 & 12,\ \ 22,\ \ 21 \\
 & \ 1\ \ \ \ \ 2\ \ \ \ \ 3 \\
k=2 & 12,\ \ 23 \\
 & \ 1\ \ \ \ \ 2 \\
k=3 & 12 \\
 & \ 1
\end{array} \tag{2.53}$$

In (2.53): for $k = 0$ units 1 and 2 are (trivially) renamed in increasing order 1 and 2; for $k = 1$, using the previously adopted new names, the units are 12,22, and 21 and are renamed in increasing order 1,2, and 3; similarly for $k = 2$, using the previously adopted new names, the units are 12 and 23 and are renamed in increasing order 1 and 2; finally for $k = 3$, using the previously adopted new names, the (just one) unit is 12 and is renamed in increasing order 1.

Since this successive renaming of the units in increasing order produces for all values of k the same new names $1, 2, \cdots$ it would be better, wishing to avoid confusion, to add lower indices to these new names denoting the value of k for which we do the renaming. For example in (2.53): for $k = 0$ units 1 and 2 may be renamed 1_0 and 2_0; for $k = 1$ units 12,22, and 21 now may be written $1_02_0, 2_02_0$, and 2_01_0 and may be renamed $1_1, 2_1$, and 3_1; and so on for the other values of k.

Writing again (2.53) in the above elucidating manner we obtain the following array.

$$\begin{array}{ll}
\multicolumn{2}{c}{units\ for\ line\ |f(1|1)|:} \\
k=0 & 1,\ \ \ 2 \\
 & 1_0\ \ 2_0 \\
k=1 & 1_02_0,\ \ 2_02_0,\ \ 2_01_0 \\
 & \ \ 1_1\ \ \ \ \ \ \ 2_1\ \ \ \ \ \ \ 3_1 \\
k=2 & 1_12_1,\ \ 2_13_1 \\
 & \ \ 1_2\ \ \ \ \ \ \ 2_2 \\
k=3 & 1_22_2 \\
 & \ \ 1_3
\end{array} \tag{2.54}$$

Next we do for lines $|f(1|1)|$, $|f(1|2)|$, and $|f(1|3)|$ what we have done for line $|f(1|1)|$ in (2.53). Of course it would be better to use the elucidating manner of (2.54) with the lower indices but since now, after the clarifications, there is no danger of confusion we prefer the simpler manner of (2.53) without the lower indices. After all, what we wish to show seems better using the manner of (2.53).

Thus in (2.55) we present the corresponding to (2.53) arrays for the three lines.

	$\lvert f(1\|1)\rvert$	$\lvert f(1\|2)\rvert$	$\lvert f(1\|3)\rvert$
$k=0$	1, 2	1, 2	1, 2
	1 2	1 2	1 2
$k=1$	12,22,21	12,22,21	12,22,21
	1 2 3	1 2 3	1 2 3
$k=2$	12, 23	11,22,33	11,22,33
	1 2	1 2 3	1 2 3
$k=3$	12	12, 23	11,22,33
	1	1 2	1 2 3
$k=4$		11, 22	12, 23
		1 2	1 2
$k=5$		12	11, 22
		1	1 2
$k=6$			11, 22
			1 2
$k=7$			12
			1

(2.55)

In (2.55) in the column of $|f(1|1)|$ we have written again (2.53). The other two columns, i.e., the columns with $|f(1|2)|$ and $|f(1|3)|$ have been produced from (2.51b) and (2.52b) respectively similarly as (2.53) has been produced from (2.50b).

Next from (2.55) we isolate the rows for which there is no explicitly expressed the value of k at the left side of them, that is, the rows in which we present the new names of the units in increasing order. We do this to show separately the new names of the units for the various values of k. This subpart of (2.55) appears in (2.56).

	$\lvert f(1\lvert 1)\rvert$	$\lvert f(1\lvert 2)\rvert$	$\lvert f(1\lvert 3)\rvert$
$k=0$	1 2	1 2	1 2
$k=1$	1 2 3	1 2 3	1 2 3
$k=2$	1 2	1 2 3	1 2 3
$k=3$	1	1 2	1 2 3
$k=4$		1 2	1 2
$k=5$		1	1 2
$k=6$			1 2
$k=7$			1

(2.56)

At the left side of (2.56) we have included the values of k. A pattern for the new names is clear in the examples of (2.56).

3 Let us express more systematically the procedure taking place in (2.50b),(2.51b),(2.52b) as well as in (2.53)–(2.56). As basic example we consider the line $|f(1|3)|$ having in mind (2.52b),(2.55),(2.56) and the notation in (2.54).

Let us write down analytically the units and their successive renamings for the case of $|f(1|3)|$ and the various values of k. The procedure now is well known from the previously given descriptions so additional explanations are not necessary.

$$\textit{units for } |f(1|3)| :$$

$$k=0: \quad \underset{1_0}{1}, \ \underset{2_0}{2} \ \in \{1,\cdots,2^{2^0}=2^1=2\} \tag{2.57a}$$

$$k=1: \quad 12,22,21= \quad 1_0 2_0, 2_0 2_0, 2_0 1_0 \equiv$$

$$\underset{1_1}{(1-1)\cdot 2^1+2}, \quad \underset{2_1}{(2-1)\cdot 2^1+2}, \quad \underset{3_1}{(2-1)\cdot 2^1+1}$$

$$=2,4,3 \in \quad \{1,\cdots,2^{2^1}=2^2=4\} \tag{2.57b}$$

$$k = 2: \qquad 22, 44, 33 = 1_1 1_1, 2_1 2_1, 3_1 3_1 \equiv$$

$$\underset{1_2}{[\,((1-1)\cdot 2^1 + 2) - 1\,] \cdot 2^2 + ((1-1)\cdot 2^1 + 2)},$$
$$\underset{2_2}{[\,((2-1)\cdot 2^1 + 2) - 1\,] \cdot 2^2 + ((2-1)\cdot 2^1 + 2)},$$
$$\underset{3_2}{[\,((2-1)\cdot 2^1 + 1) - 1\,] \cdot 2^2 + ((2-1)\cdot 2^1 + 1)}$$

$$= 6, 16, 11 \in \{1, \cdots, 2^{2^2} = 2^4 = 16\} \tag{2.57c}$$

$$k = 3: \qquad 6\;6, 16\;16, 11\;11 = 1_2 1_2, 2_2 2_2, 3_2 3_2 \equiv$$

$$\underset{1_3}{(\{[\,((1-1)\cdot 2^1 + 2) - 1\,] \cdot 2^2 + ((1-1)\cdot 2^1 + 2)\} - 1) \cdot 2^4 + \{[\,((1-1)\cdot 2^1 + 2) - 1\,] \cdot 2^2 + ((1-1)\cdot 2^1 + 2)\}},$$
$$\underset{2_3}{(\{[\,((2-1)\cdot 2^1 + 2) - 1\,] \cdot 2^2 + ((2-1)\cdot 2^1 + 2)\} - 1) \cdot 2^4 + \{[\,((2-1)\cdot 2^1 + 2) - 1\,] \cdot 2^2 + ((2-1)\cdot 2^1 + 2)\}},$$
$$\underset{3_3}{(\{[\,((2-1)\cdot 2^1 + 1) - 1\,] \cdot 2^2 + ((2-1)\cdot 2^1 + 1)\} - 1) \cdot 2^4 + \{[\,((2-1)\cdot 2^1 + 1) - 1\,] \cdot 2^2 + ((2-1)\cdot 2^1 + 1)\}}$$

$$= 86, 256, 171 \in \{1, \cdots, 2^{2^3} = 2^8 = 256\} \tag{2.57d}$$

(we continue in a manner more brief)

$$k = 4: \qquad 86\;256, 256\;171 = \qquad 1_3 2_3, 2_3 3_3 \equiv$$

$$\underset{1_4}{(1_3 - 1)\cdot 2^8 + 2_3}, \qquad \underset{2_4}{(2_3 - 1)\cdot 2^8 + 3_3} \tag{2.57e}$$

$$\in \{1, \cdots, 2^{2^4} = 2^{16}\}$$

$$k=5: \qquad 1_4 1_4, 2_4 2_4 \equiv$$

$$\underset{1_5}{(1_4-1)\cdot 2^{16}+1_4}, \qquad \underset{2_5}{(2_4-1)\cdot 2^{16}+2_4} \tag{2.57f}$$

$$\in \{1,\cdots,2^{2^5}=2^{32}\}$$

$$k=6: \qquad 1_5 1_5, 2_5 2_5 \equiv$$

$$\underset{1_6}{(1_5-1)\cdot 2^{32}+1_5}, \qquad \underset{2_6}{(2_5-1)\cdot 2^{32}+2_5} \tag{2.57g}$$

$$\in \{1,\cdots,2^{2^6}=2^{64}\}$$

$$k=7: \qquad 1_6 2_6 \equiv$$

$$\underset{1_7}{(1_6-1)\cdot 2^{64}+2_6} \tag{2.57h}$$

$$\in \{1,\cdots,2^{2^7}=2^{128}\}$$

4 Now we write down again (2.57), for $|f(1|3)|$, in a more compact style. Similarly, in the same style, we write the examples of $|f(1|1)|$ and $|f(1|2)|$. For producing the last two examples we have again in mind (2.50b),(2.51b),(2.52b) and (2.53)–(2.56) as we did before for producing the example of $|f(1|3)|$.

$$units\ for\ |f(1|3)|:$$

$$k=0: \qquad 1_0, 2_0 \equiv 1,2 \quad \in \{1,2,\cdots,2^{2^0}=2^1=2\} \tag{2.58a}$$

$$k=1: \qquad 1_0 2_0, 2_0 2_0, 2_0 1_0 \equiv$$

$$\underset{1_1}{(1_0-1)\cdot 2^1+2_0}, \quad \underset{2_1}{(2_0-1)\cdot 2^1+2_0}, \quad \underset{3_1}{(2_0-1)\cdot 2^1+1_0}$$

$$\in \{1,2,\cdots,2^{2^1}=2^2=4\} \tag{2.58b}$$

$$k = 2: \quad 1_1 1_1, 2_1 2_1, 3_1 3_1 \equiv$$

$$\underbrace{(1_1 - 1) \cdot 2^2 + 1_1}_{1_2}, \quad \underbrace{(2_1 - 1) \cdot 2^2 + 2_1}_{2_2}, \quad \underbrace{(3_1 - 1) \cdot 2^2 + 3_1}_{3_2}$$

$$\in \{1, 2, \cdots, 2^{2^2} = 2^4 = 16\} \tag{2.58c}$$

$$k = 3: \quad 1_2 1_2, 2_2 2_2, 3_2 3_2 \equiv$$

$$\underbrace{(1_2 - 1) \cdot 2^4 + 1_2}_{1_3}, \quad \underbrace{(2_2 - 1) \cdot 2^4 + 2_2}_{2_3}, \quad \underbrace{(3_2 - 1) \cdot 2^4 + 3_2}_{3_3}$$

$$\in \{1, 2, \cdots, 2^{2^3} = 2^8 = 256\} \tag{2.58d}$$

$$k = 4: \quad 1_3 2_3, 2_3 3_3 \equiv$$

$$\underbrace{(1_3 - 1) \cdot 2^8 + 2_3}_{1_4}, \quad \underbrace{(2_3 - 1) \cdot 2^8 + 3_3}_{2_4} \tag{2.58e}$$

$$\in \{1, 2, \cdots, 2^{2^4} = 2^{16}\}$$

$$k = 5: \quad 1_4 1_4, 2_4 2_4 \equiv$$

$$\underbrace{(1_4 - 1) \cdot 2^{16} + 1_4}_{1_5}, \quad \underbrace{(2_4 - 1) \cdot 2^{16} + 2_4}_{2_5} \tag{2.58f}$$

$$\in \{1, 2, \cdots, 2^{2^5} = 2^{32}\}$$

$$k = 6: \quad 1_5 1_5, 2_5 2_5 \equiv$$

$$\underbrace{(1_5 - 1) \cdot 2^{32} + 1_5}_{1_6}, \quad \underbrace{(2_5 - 1) \cdot 2^{32} + 2_5}_{2_6} \tag{2.58g}$$

$$\in \{1, 2, \cdots, 2^{2^6} = 2^{64}\}$$

$$k = 7: \qquad 1_6 2_6 \equiv$$

$$\underset{1_7}{(1_6 - 1) \cdot 2^{64} + 2_6} \tag{2.58h}$$

$$\in \{1, 2, \cdots, 2^{2^7} = 2^{128}\}$$

---

units for $|f(1|2)|$:

$$k = 0: \qquad 1_0, 2_0 \equiv 1, 2 \quad \in \{1, 2, \cdots, 2^{2^0} = 2^1 = 2\} \tag{2.59a}$$

$$k = 1: \qquad 1_0 2_0, 2_0 2_0, 2_0 1_0 \equiv$$

$$\underset{1_1}{(1_0 - 1) \cdot 2^1 + 2_0}, \quad \underset{2_1}{(2_0 - 1) \cdot 2^1 + 2_0}, \quad \underset{3_1}{(2_0 - 1) \cdot 2^1 + 1_0} \tag{2.59b}$$

$$\in \{1, 2, \cdots, 2^{2^1} = 2^2 = 4\}$$

$$k = 2: \qquad 1_1 1_1, 2_1 2_1, 3_1 3_1 \equiv$$

$$\underset{1_2}{(1_1 - 1) \cdot 2^2 + 1_1}, \quad \underset{2_2}{(2_1 - 1) \cdot 2^2 + 2_1}, \quad \underset{3_2}{(3_1 - 1) \cdot 2^2 + 3_1} \tag{2.59c}$$

$$\in \{1, 2, \cdots, 2^{2^2} = 2^4 = 16\}$$

$$k = 3: \qquad 1_2 2_2, 2_2 3_2 \equiv$$

$$\underset{1_3}{(1_2 - 1) \cdot 2^4 + 2_2}, \qquad \underset{2_3}{(2_2 - 1) \cdot 2^4 + 3_2}$$

$$\in \{1, 2, \cdots, 2^{2^3} = 2^8 = 256\} \tag{2.59d}$$

$$k = 4: \qquad 1_3 1_3, 2_3 2_3 \equiv$$

$$\underset{1_4}{(1_3 - 1) \cdot 2^8 + 1_3}, \qquad \underset{2_4}{(2_3 - 1) \cdot 2^8 + 2_3} \tag{2.59e}$$

$$\in \{1, 2, \cdots, 2^{2^4} = 2^{16}\}$$

$$k = 5: \qquad 1_4 2_4 \equiv$$

$$\underset{1_5}{(1_4 - 1) \cdot 2^{16} + 2_4} \tag{2.59f}$$

$$\in \{1, 2, \cdots, 2^{2^5} = 2^{32}\}$$

$$units\ for\ |f(1|1)|:$$

$$k = 0: \qquad 1_0, 2_0 \equiv 1, 2 \quad \in \{1, 2, \cdots, 2^{2^0} = 2^1 = 2\} \tag{2.60a}$$

$$k = 1: \qquad 1_0 2_0, 2_0 2_0, 2_0 1_0 \equiv$$

$$\underset{1_1}{(1_0 - 1) \cdot 2^1 + 2_0}, \quad \underset{2_1}{(2_0 - 1) \cdot 2^1 + 2_0}, \quad \underset{3_1}{(2_0 - 1) \cdot 2^1 + 1_0} \tag{2.60b}$$

$$\in \{1, 2, \cdots, 2^{2^1} = 2^2 = 4\}$$

$$k = 2: \qquad 1_1 2_1, 2_1 3_1 \equiv$$

$$\underset{1_2}{(1_1 - 1) \cdot 2^2 + 2_1}, \qquad \underset{2_2}{(2_1 - 1) \cdot 2^2 + 3_1}$$

$$\in \{1, 2, \cdots, 2^{2^2} = 2^4 = 16\} \tag{2.60c}$$

$$k = 3: \qquad 1_2 2_2 \equiv$$

$$\begin{matrix} (1_2 - 1) \cdot 2^4 + 2_2 \\ 1_3 \end{matrix} \tag{2.60d}$$

$$\in \{1, 2, \cdots, 2^{2^3} = 2^8 = 256\}$$

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5 Having produced in (2.58)–(2.60) the units for the lines $|f(1|1)|$, $|f(1|2)|$, and $|f(1|3)|$ we can similarly produce the units for the line $|f(1|j)|$ which is the general case. This can be done in a direct manner taking into account the general form of the lines $|f(i|j)|$ [which are the lines $2P^{\,i}_{\,j}$] as it is described especially in Chapter 5 of the *Mathematical Diary 1993–1998* and also in Chapters 6 and 7 of the same work. The examples in (2.38)–(2.40) are also of help. Next we present directly the units for the line $|f(1|j)|$ in exactly the same style we have seen in the examples (2.58)–(2.60).

$$units\ for\ |f(1|j)|:$$

$$the\ values\ of\ k:$$

$$k \in \{0; 1, 2, \cdots, j; j+1, j+2, \cdots, 2j; 2j+1\} \tag{2.61}$$

or better

$$\begin{matrix} k \in \{ & 0, & \\ & 1, 2, \cdots, j, & \\ & j+1, j+2, \cdots, 2j, & \\ & 2j+1 & \} \end{matrix} \tag{2.62}$$

$$the\ units:$$

$$k = 0: \qquad 1_k, 2_k \equiv 1, 2 \quad \in \{1, 2, \cdots, 2^{2^k}\} \tag{2.63a}$$

$$k = 1: \qquad 1_{k-1}2_{k-1}, 2_{k-1}2_{k-1}, 2_{k-1}1_{k-1} \equiv$$

$$\underset{1_k}{(1_{k-1}-1)\cdot 2^{2^{k-1}}+2_{k-1}}, \quad \underset{2_k}{(2_{k-1}-1)\cdot 2^{2^{k-1}}+2_{k-1}}, \quad \underset{3_k}{(2_{k-1}-1)\cdot 2^{2^{k-1}}+1_{k-1}} \tag{2.63b}$$

$$\in \{1, 2, \cdots, 2^{2^k}\}$$

$$2 \leqslant k \leqslant j: \qquad 1_{k-1}1_{k-1}, 2_{k-1}2_{k-1}, 3_{k-1}3_{k-1} \equiv$$

$$\underset{1_k}{(1_{k-1}-1)\cdot 2^{2^{k-1}}+1_{k-1}}, \quad \underset{2_k}{(2_{k-1}-1)\cdot 2^{2^{k-1}}+2_{k-1}}, \quad \underset{3_k}{(3_{k-1}-1)\cdot 2^{2^{k-1}}+3_{k-1}} \tag{2.63c}$$

$$\in \{1, 2, \cdots, 2^{2^k}\}$$

$$k = j+1: \qquad 1_{k-1}2_{k-1}, 2_{k-1}3_{k-1} \equiv$$

$$\underset{1_k}{(1_{k-1}-1)\cdot 2^{2^{k-1}}+2_{k-1}}, \quad \underset{2_k}{(2_{k-1}-1)\cdot 2^{2^{k-1}}+3_{k-1}}$$

$$\in \{1, 2, \cdots, 2^{2^k}\} \tag{2.63d}$$

$$j+2 \leqslant k \leqslant 2j: \qquad 1_{k-1}1_{k-1}, 2_{k-1}2_{k-1} \equiv$$

$$\underset{1_k}{(1_{k-1}-1)\cdot 2^{2^{k-1}}+1_{k-1}}, \quad \underset{2_k}{(2_{k-1}-1)\cdot 2^{2^{k-1}}+2_{k-1}}$$

$$\in \{1, 2, \cdots, 2^{2^k}\} \tag{2.63e}$$

$$k = 2j+1: \qquad \underset{1_k}{1_{k-1}2_{k-1} \equiv (1_{k-1}-1)\cdot 2^{2^{k-1}} + 2_{k-1}} \in \{1, 2, \cdots, 2^{2^k}\} \tag{2.63f}$$

6 Next, in Table 2.3, we write again (2.61)–(2.63) in a form more compact.

Table 2.3 Recursive formulae by which the basic units, that compose the lines $|f(1|j)|$, are determined for the various cases k.

$$Units\ for\ |f(1|j)|:$$

$$The\ entities\ k, \alpha, \beta:$$

$$k \in \{0, 1, 2, \cdots, 2j+1\} \tag{2.64}$$

For α and β, each of which takes values from the set $\{1, 2, 3\}$, and for any k it is

$$\alpha_{k-1}\beta_{k-1} \equiv (\alpha_{k-1}-1)\cdot 2^{2^{k-1}} + \beta_{k-1} \in \{1, 2, \cdots, 2^{2^k}\} \tag{2.65}$$

$$The\ units:$$

$$k = 0: \qquad \underset{1_k}{1}, \quad \underset{2_k}{2} \quad \in \{1, 2, \cdots, 2^{2^k}\} \tag{2.66a}$$

$$k = 1: \qquad \underset{1_k}{1_{k-1}2_{k-1}}, \quad \underset{2_k}{2_{k-1}2_{k-1}}, \quad \underset{3_k}{2_{k-1}1_{k-1}} \tag{2.66b}$$

$$2 \leqslant k \leqslant j: \qquad \underset{1_k}{1_{k-1}1_{k-1}}, \quad \underset{2_k}{2_{k-1}2_{k-1}}, \quad \underset{3_k}{3_{k-1}3_{k-1}} \tag{2.66c}$$

$$k = j + 1 : \quad \begin{matrix} 1_{k-1}2_{k-1}, & 2_{k-1}3_{k-1} \\ 1_k & 2_k \end{matrix} \tag{2.66d}$$

$$j + 2 \leqslant k \leqslant 2j : \quad \begin{matrix} 1_{k-1}1_{k-1}, & 2_{k-1}2_{k-1} \\ 1_k & 2_k \end{matrix} \tag{2.66e}$$

$$k = 2j + 1 : \quad \begin{matrix} 1_{k-1}2_{k-1} \\ 1_k \end{matrix} \tag{2.66f}$$

The content of Table 2.3 is the main result of the present Section 2.4.

2.5 Expressing the units of the algorithm of Table 2.3 in a nonrecursive manner.

1 For the applications perhaps it is better expressing the units, appearing in Table 2.3, with the help of *nonrecursive* formulae for the various values of k. Let us see it analytically.

For producing nonrecursive formulae we prefer to use the convention (2.29) or (2.30) for the strings l instead of the convention (2.28), that is, we accept

$$l \in \{0, 1, 2, \cdots, 2^{2^k} - 1\}. \tag{2.67}$$

Then, instead of the relation

$$\alpha_{k-1}\beta_{k-1} \equiv (\alpha_{k-1} - 1) \cdot 2^{2^{k-1}} + \beta_{k-1}, \tag{2.68}$$

appearing in (2.65), we have the simpler, and more convenient for constructing nonrecursive formulae, relation

$$\alpha_{k-1}\beta_{k-1} \equiv \alpha_{k-1} \cdot 2^{2^{k-1}} + \beta_{k-1}. \tag{2.69}$$

Note 2.4 We insist again that using examples is important for clarifying the formulae. So for understanding the difference between (2.68) and convention (2.28) on the one side and (2.69) and convention (2.67) on the other side it is enough, beyond theoretical descriptions, to consider an elementary example from Section 1.1. Having in mind the renamings (1.1),(1.4),(1.5) of the strings we can see directly that:(i) According to the convention (2.28) first we have strings 1,2 that next produce strings 11,12,21,22 which are renamed 1,2,3,4 respectively; thus string e.g. 21 takes new name 3 and indeed from (2.68) we see that $21 = (2-1) \cdot 2 + 1 = 3$. (ii) According to the convention (2.67) first we have strings 0,1 that next produce strings 00,01,10,11 which are renamed 0,1,2,3 respectively; thus string e.g. 10 takes new name 2 and indeed from (2.69) we see that $10 = 1 \cdot 2 + 0 = 2$. $\square$

Thus using the convention (2.67), instead of (2.28), Table 2.3 [which is constructed in accordance with convention (2.28)] takes the slightly simpler form of Table 2.5.

Table 2.5 A slightly simpler form of Table 2.3 based on convention (2.67) and relation (2.69) [instead of convention (2.28) and relation (2.68) on which Table 2.3 was based].

$$Units\ for\ |f(1|j)|:$$

$$The\ entities\ k, \alpha, \beta:$$

$$k \in \{0, 1, 2, \cdots, 2j+1\} \tag{2.70}$$

For α and β, each of which takes values from the set $\{1, 2, 3\}$, and for any k it is

$$\alpha_{k-1}\beta_{k-1} \equiv \alpha_{k-1} \cdot 2^{2^{k-1}} + \beta_{k-1} \in \{0, 1, 2, \cdots, 2^{2^k} - 1\} \tag{2.71}$$

$$The\ units:$$

$$k = 0: \qquad \begin{matrix} 0, & 1 \\ 1_k & 2_k \end{matrix} \quad \in \{0, 1, 2, \cdots, 2^{2^k} - 1\} \tag{2.72a}$$

$$k = 1: \qquad \begin{matrix} 1_{k-1}2_{k-1}, & 2_{k-1}2_{k-1}, & 2_{k-1}1_{k-1} \\ 1_k & 2_k & 3_k \end{matrix} \tag{2.72b}$$

$$2 \leqslant k \leqslant j : \quad \begin{matrix} 1_{k-1}1_{k-1}, & 2_{k-1}2_{k-1}, & 3_{k-1}3_{k-1} \\ 1_k & 2_k & 3_k \end{matrix} \tag{2.72c}$$

$$k = j+1 : \quad \begin{matrix} 1_{k-1}2_{k-1}, & 2_{k-1}3_{k-1} \\ 1_k & 2_k \end{matrix} \tag{2.72d}$$

$$j+2 \leqslant k \leqslant 2j : \quad \begin{matrix} 1_{k-1}1_{k-1}, & 2_{k-1}2_{k-1} \\ 1_k & 2_k \end{matrix} \tag{2.72e}$$

$$k = 2j+1 : \quad \begin{matrix} 1_{k-1}2_{k-1} \\ 1_k \end{matrix} \tag{2.72f}$$

---

2 Let us write (2.72) using equivalent *nonrecursive* formulae.

For this purpose we consider as example the case with $j = 3$, that is, $|f(1|3)|$. Using Table 2.5 and especially the recursive formulae (2.72), but also formula (2.71), we obtain the units for the various values of k in a form nonrecursive. This means that instead of expressing the units for k with respect to the units for $k-1$, as is the case in (2.72) of Table 2.5, we are going to express them in a totally elementary form independent of the units for the previous values of k.

The results for the example $|f(1|3)|$ are presented in the following without further descriptions.

$$Units\ for\ |f(1|3)| :$$

$$the\ values\ of\ k :$$

$$k \in \{0, 1, 2, \cdots, 7\} \tag{2.73}$$

$$the\ units :$$

$$k=0: \qquad \underset{1_0}{0}, \quad \underset{2_0}{1} \quad \in \{0,1\} \tag{2.74a}$$

$$\begin{aligned} k=1: \qquad 1_0 2_0 \equiv 1_0 \cdot 2^{2^0} + 2_0 &= 0 \cdot 2^1 + 1 \\ = 1 &= \underline{1} \qquad \equiv 1_1 \\ 2_0 2_0 \equiv 2_0 \cdot 2^{2^0} + 2_0 &= 1 \cdot 2^1 + 1 \\ = 3 &= \underline{2^{2^0} + 1} \qquad \equiv 2_1 \\ 2_0 1_0 \equiv 2_0 \cdot 2^{2^0} + 1_0 &= 1 \cdot 2^1 + 0 \\ = 2 &= \underline{2^{2^0}} \qquad \equiv 3_1 \end{aligned} \tag{2.74b}$$

$$\begin{aligned} k=2: \quad 1_1 1_1 \equiv 1_1 \cdot 2^{2^1} + 1_1 = &\quad (1_0 \cdot 2^{2^0} + 2_0) \cdot 2^{2^1} + (1_0 \cdot 2^{2^0} + 2_0) \\ &= (1_0 \cdot 2^{2^0} + 2_0) \cdot (2^{2^1} + 1) \\ &= 1 \cdot (2^{2^1} + 1) = \underline{2^{2^1} + 1} \equiv 1_2 \\ 2_1 2_1 \equiv 2_1 \cdot 2^{2^1} + 2_1 = &\quad (2_0 \cdot 2^{2^0} + 2_0) \cdot 2^{2^1} + (2_0 \cdot 2^{2^0} + 2_0) \\ &= (2_0 \cdot 2^{2^0} + 2_0) \cdot (2^{2^1} + 1) \\ &= \underline{(2^{2^0} + 1) \cdot (2^{2^1} + 1)} \equiv 2_2 \\ 3_1 3_1 \equiv 3_1 \cdot 2^{2^1} + 3_1 = &\quad (2_0 \cdot 2^{2^0} + 1_0) \cdot 2^{2^1} + (2_0 \cdot 2^{2^0} + 1_0) \\ &= (2_0 \cdot 2^{2^0} + 1_0) \cdot (2^{2^1} + 1) \\ &= \underline{2^{2^0} \cdot (2^{2^1} + 1)} \equiv 3_2 \end{aligned} \tag{2.74c}$$

$$
\begin{aligned}
k=3: \quad & 1_2 1_2 \equiv 1_2 \cdot 2^{2^2} + 1_2 \\
& = 1_2 \cdot (2^{2^2}+1) = \qquad \underline{(2^{2^1}+1)\cdot(2^{2^2}+1)} \equiv 1_3 \\
& 2_2 2_2 \equiv 2_2 \cdot 2^{2^2} + 2_2 \\
& = 2_2 \cdot (2^{2^2}+1) = \quad \underline{(2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^2}+1)} \equiv 2_3 \\
& 3_2 3_2 \equiv 3_2 \cdot 2^{2^2} + 3_2 \\
& = 3_2 \cdot (2^{2^2}+1) = \qquad \underline{2^{2^0}\cdot(2^{2^1}+1)\cdot(2^{2^2}+1)} \equiv 3_3
\end{aligned} \tag{2.74d}
$$

$$
\begin{aligned}
k=4: \quad & 1_3 2_3 \equiv 1_3 \cdot 2^{2^3} + 2_3 \\
& = (2^{2^1}+1)\cdot(2^{2^2}+1)\cdot 2^{2^3} + (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^2}+1) \\
& = \underline{(2^{2^1}+1)\cdot(2^{2^2}+1)\cdot[\,2^{2^3}+2^{2^0}+1\,]} \equiv 1_4 \\
& 2_3 3_3 \equiv 2_3 \cdot 2^{2^3} + 3_3 = \\
& (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^2}+1)\cdot 2^{2^3} + 2^{2^0}\cdot(2^{2^1}+1)\cdot(2^{2^2}+1) \\
& = (2^{2^1}+1)\cdot(2^{2^2}+1)\cdot[\,(2^{2^0}+1)\cdot 2^{2^3} + 2^{2^0}\,] \\
& = \underline{(2^{2^1}+1)\cdot(2^{2^2}+1)\cdot[\,2^{2^3+2^0} + 2^{2^3} + 2^{2^0}\,]} \equiv 2_4
\end{aligned} \tag{2.74e}
$$

$$k=5:\quad 1_4 1_4 \equiv 1_4\cdot 2^{2^4}+1_4 = 1_4\cdot(2^{2^4}+1)$$

$$=\underline{(2^{2^1}+1)\cdot(2^{2^2}+1)\cdot[\,2^{2^3}+2^{2^0}+1\,]\cdot(2^{2^4}+1)}\equiv 1_5$$

$$2_4 2_4 \equiv 2_4\cdot 2^{2^4}+2_4 = 2_4\cdot(2^{2^4}+1)$$

$$=\underline{(2^{2^1}+1)\cdot(2^{2^2}+1)\cdot[\,2^{2^3+2^0}+2^{2^3}+2^{2^0}\,]\cdot(2^{2^4}+1)}\equiv 2_5 \tag{2.74f}$$

$$k=6:\quad 1_5 1_5 \equiv 1_5\cdot 2^{2^5}+1_5 = 1_5\cdot(2^{2^5}+1)$$

$$=\underline{(2^{2^1}+1)\cdot(2^{2^2}+1)}$$

$$\underline{\cdot[\,2^{2^3}+2^{2^0}+1\,]\cdot(2^{2^4}+1)\cdot(2^{2^5}+1)}\equiv 1_6$$

$$2_5 2_5 \equiv 2_5\cdot 2^{2^5}+2_5 = 2_5\cdot(2^{2^5}+1)$$

$$=\underline{(2^{2^1}+1)\cdot(2^{2^2}+1)}$$

$$\underline{\cdot[\,2^{2^3+2^0}+2^{2^3}+2^{2^0}\,]\cdot(2^{2^4}+1)\cdot(2^{2^5}+1)}\equiv 2_6 \tag{2.74g}$$

$$k=7:\quad 1_6 2_6 \equiv 1_6\cdot 2^{2^6}+2_6 = \underline{(2^{2^1}+1)\cdot(2^{2^2}+1)}$$

$$\underline{\cdot[\,2^{2^6+2^3}+2^{2^6+2^0}+2^{2^3+2^0}+2^{2^6}+2^{2^3}+2^{2^0}\,]} \tag{2.74h}$$

$$\underline{\cdot(2^{2^4}+1)\cdot(2^{2^5}+1)}\equiv 1_7$$

In (2.74) we have underlined the basic results which are the basic units expressed in nonrecursive form. For example, in (2.74h) the basic unit for $k=7$, which is 1_7 [and which in (2.72f) of Table 2.5 is expressed recursively as $1_6 2_6$], is expressed as the underlined product of factors. This product is in a form nonrecursive.

3 As we did, in (2.73)–(2.74), the example with $j=3$, that is, the example of $|f(1|3)|$ we do next the general case with random j, that is, the case of $|f(1|j)|$. The result is presented, without further comments, in Table 2.6 which is equivalent to Table 2.5 but with the basic units now expressed in *nonrecursive* form.

Table 2.6 A table equivalent to Table 2.5 but with the basic units expressed in nonrecursive form.

$$Units\ for\ |f(1|j)|\ (nonrecursively):$$

$$the\ values\ of\ k:$$

$$k \in \{0, 1, 2, \cdots, 2j+1\} \tag{2.75}$$

$$the\ units:$$

$$k=0: \quad \begin{array}{lll} 0 & \equiv 1_0 & = 1_k \\ 1 & \equiv 2_0 & = 2_k \end{array} \tag{2.76a}$$

$$k=1: \quad \begin{array}{llll} 1_0 2_0 \equiv 1_0 \cdot 2^{2^0} + 2_0 & = 1 & \equiv 1_1 = 1_k \\ 2_0 2_0 \equiv 2_0 \cdot 2^{2^0} + 2_0 & = 2^{2^0} + 1 & \equiv 2_1 = 2_k \\ 2_0 1_0 \equiv 2_0 \cdot 2^{2^0} + 1_0 & = 2^{2^0} & \equiv 3_1 = 3_k \end{array} \tag{2.76b}$$

$$
\begin{aligned}
2 \leqslant k \leqslant j: \quad & 1_{k-1} 1_{k-1} \equiv \\
& (2^{2^1}+1)\cdot(2^{2^2}+1)\cdots(2^{2^{k-1}}+1) && \equiv 1_k \\
& 2_{k-1} 2_{k-1} \equiv \\
& (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^2}+1)\cdots(2^{2^{k-1}}+1) && \equiv 2_k \\
& 3_{k-1} 3_{k-1} \equiv \\
& 2^{2^0}\cdot(2^{2^1}+1)\cdot(2^{2^2}+1)\cdots(2^{2^{k-1}}+1) && \equiv 3_k
\end{aligned}
\tag{2.76c}
$$

$$
\begin{aligned}
k = j+1: \quad & 1_j 2_j \equiv \\
& (2^{2^1}+1)\cdot(2^{2^2}+1)\cdots(2^{2^{j-1}}+1) \\
& \cdot[\, 2^{2^j} + 2^{2^0} + 1 \,] && \equiv 1_{j+1} = 1_k \\
& 2_j 3_j \equiv \\
& (2^{2^1}+1)\cdot(2^{2^2}+1)\cdots(2^{2^{j-1}}+1) \\
& \cdot[\, 2^{2^j+2^0} + 2^{2^j} + 2^{2^0} \,] && \equiv 2_{j+1} = 2_k
\end{aligned}
\tag{2.76d}
$$

$$j+2 \leqslant k \leqslant 2j: \quad 1_{k-1}1_{k-1} \equiv$$

$$(2^{2^1}+1)\cdot(2^{2^2}+1)\cdots(2^{2^{j-1}}+1)$$

$$\cdot[\,2^{2^j}+2^{2^0}+1\,]$$

$$\cdot(2^{2^{j+1}}+1)\cdot(2^{2^{j+2}}+1)\cdots(2^{2^{k-1}}+1) \quad \equiv 1_k$$

$$2_{k-1}2_{k-1} \equiv$$

$$(2^{2^1}+1)\cdot(2^{2^2}+1)\cdots(2^{2^{j-1}}+1)$$

$$\cdot[\,2^{2^j+2^0}+2^{2^j}+2^{2^0}\,]$$

$$\cdot(2^{2^{j+1}}+1)\cdot(2^{2^{j+2}}+1)\cdots(2^{2^{k-1}}+1) \quad \equiv 2_k \tag{2.76e}$$

$$k=2j+1: \quad 1_{k-1}2_{k-1} \equiv$$

$$(2^{2^1}+1)\cdot(2^{2^2}+1)\cdots(2^{2^{j-1}}+1)$$

$$\cdot[\,2^{2^{2j}+2^j}+2^{2^{2j}+2^0}+2^{2^j+2^0}+2^{2^{2j}}+2^{2^j}+2^{2^0}\,]$$

$$\cdot(2^{2^{j+1}}+1)\cdot(2^{2^{j+2}}+1)\cdots(2^{2^{2j-1}}+1) \equiv 1_{2j+1} \quad = 1_k \tag{2.76f}$$

---

4 Next we present Table 2.7 which is a compactified version of Table 2.6. Since all is evident there is no need of additional descriptions.

Table 2.7 A more compact form of Table 2.6.

$$Units\ for\ |f(1|j)|\ (nonrecursively):$$

$$\textit{the values of } k:$$

$$k \in \{0, 1, 2, \cdots, 2j+1\} \tag{2.77}$$

$$\textit{various entities}:$$

$$A_m \equiv (2^{2^m} + 1) \tag{2.78}$$

$$\begin{aligned} X &\equiv (2^{2^j} + 2^{2^0} + 1) \\ Y &\equiv (2^{2^j + 2^0} + 2^{2^j} + 2^{2^0}) \\ Z &\equiv (2^{2^{2j} + 2^j} + 2^{2^{2j} + 2^0} + 2^{2^j + 2^0} + 2^{2^{2j}} + 2^{2^j} + 2^{2^0}) \end{aligned} \tag{2.79}$$

$$\begin{aligned} &(1_0, 2_0) \equiv (0, 1) \\ &(1_1, 2_1, 3_1) \equiv (1, 2^{2^0} + 1, 2^{2^0}) = (1, 3, 2) \end{aligned} \tag{2.80}$$

$$\textit{the units}:$$

$$k = 0: \qquad (1_k, 2_k) = (1_0, 2_0) \tag{2.81a}$$

$$k = 1: \qquad (1_k, 2_k, 3_k) = (1_1, 2_1, 3_1) \tag{2.81b}$$

$$\begin{aligned} 2 \leqslant k \leqslant j: \quad & 1_k \equiv 1_1 \cdot A_1 \cdot A_2 \cdots A_{k-1} \\ & 2_k \equiv 2_1 \cdot A_1 \cdot A_2 \cdots A_{k-1} \\ & 3_k \equiv 3_1 \cdot A_1 \cdot A_2 \cdots A_{k-1} \end{aligned} \tag{2.81c}$$

$$k = j+1: \quad \begin{aligned} 1_k &= 1_{j+1} \equiv A_1 \cdot A_2 \cdots A_{j-1} \cdot X \\ 2_k &= 2_{j+1} \equiv A_1 \cdot A_2 \cdots A_{j-1} \cdot Y \end{aligned} \tag{2.81d}$$

$$j+2 \leqslant k \leqslant 2j: \quad \begin{aligned} 1_k &\equiv A_1 \cdot A_2 \cdots A_{j-1} \cdot X \cdot A_{j+1} \cdot A_{j+2} \cdots A_{k-1} \\ 2_k &\equiv A_1 \cdot A_2 \cdots A_{j-1} \cdot Y \cdot A_{j+1} \cdot A_{j+2} \cdots A_{k-1} \end{aligned} \tag{2.81e}$$

$$k = 2j+1: \quad 1_k = 1_{2j+1} \equiv A_1 \cdot A_2 \cdots A_{j-1} \cdot Z \cdot A_{j+1} \cdot A_{j+2} \cdots A_{2j-1} \tag{2.81f}$$

5 Writing Table 2.7 even more compactly we obtain the equivalent Table 2.8.

Table 2.8 A more compact version of Table 2.7.

$$Units\ for\ |f(1|j)|\ (nonrecursively):$$

All the entities $k, A_m, X, Y, Z, (1_0, 2_0), (1_1, 2_1, 3_1)$ are exactly as in Table 2.7

For $m \neq j$ it is $B_m \equiv A_m$

For $m = j$ it is $B_m \equiv X$ or Y or Z according to whether B_m is in the scope of the product symbol $\mathtt{X}\prod$ or $\mathtt{Y}\prod$ or $\mathtt{Z}\prod$ respectively

$$the\ units:$$

$$k = 0: \quad (1_k, 2_k) \equiv (1_0, 2_0) \tag{2.82a}$$

$$k = 1: \qquad (1_k, 2_k, 3_k) \equiv (1_1, 2_1, 3_1) \tag{2.82b}$$

$$2 \leqslant k \leqslant j: \qquad (1_k, 2_k, 3_k) \equiv (1_1, 2_1, 3_1) \cdot \left(\prod_{m=1}^{k-1} B_m \right) \tag{2.82c}$$

$$\left.\begin{array}{rl} k = j+1: & (1_k, 2_k) \equiv \left[\left(\mathsf{X} \prod_{m=1}^{k-1} B_m \right), \left(\mathsf{Y} \prod_{m=1}^{k-1} B_m \right) \right] \\ j+2 \leqslant k \leqslant 2j: & (1_k, 2_k) \equiv \left[\left(\mathsf{X} \prod_{m=1}^{k-1} B_m \right), \left(\mathsf{Y} \prod_{m=1}^{k-1} B_m \right) \right] \end{array} \right\} \text{same form} \tag{2.82d}$$

$$k = 2j+1: \qquad 1_k \equiv \left(\mathsf{Z} \prod_{m=1}^{2j-1} B_m \right) \tag{2.82e}$$

In (2.82d) the form for the units is same for both $k = j + 1$ and $j+2 \leqslant k \leqslant 2j$ thus it could be written just once for $j+1 \leqslant k \leqslant 2j$. But we prefer to write it twice because in the previous equivalent tables we have written cases $k = j+1$ and $j+2 \leqslant k \leqslant 2j$ separately (there the forms of the units were different for these two cases).

2.6 Determining how the lines $|f(1|j)|$ are composed of the basic units $1_k, 2_k, 3_k$.

1 Next with the help of the examples (2.50)–(2.52) and using now the new names $1_k, 2_k, 3_k$ for the basic units we determine how the lines $|f(1|j)|$ are composed, as a whole, analytically of these basic units.

First we present the examples of the lines $|f(1|j)|$ for $j = 1, 2, 3$. The basic units are named $1_k, 2_k, 3_k$ and all is clear from the previous descriptions. By "period", as usual, we mean the length of the strings. Instead of writing, e.g., "$|f(1|j)|$ for $k = \alpha$" we just write

$|f(1|j)|_{k=\alpha}$ or $|f(1|j)|_{\alpha}$.

$$
\begin{aligned}
&(j=1, \quad period\ \ 2^3)\\
|f(1|1)|_{k=0} =&\ \ 1_0 2_0 2_0 2_0 2_0 2_0 2_0 1_0\\
|f(1|1)|_{k=1} =&\ \ 1_1 2_1 2_1 3_1\\
|f(1|1)|_{k=2} =&\ \ 1_2 2_2\\
|f(1|1)|_{k=3} =&\ \ 1_3
\end{aligned}
\tag{2.83}
$$

$$
\begin{aligned}
&(j=2, \quad period\ \ 2^5)\\
|f(1|2)|_{k=0} =&\ \ 1_0 2_0 1_0 2_0 2_0 2_0 2_0 2_0 1_0 2_0 1_0 2_0 2_0 2_0 2_0 2_0\\
&\ \ 2_0 2_0 2_0 2_0 2_0 1_0 2_0 1_0 2_0 2_0 2_0 2_0 2_0 1_0 2_0 1_0\\
|f(1|2)|_{k=1} =&\ \ 1_1 1_1 2_1 2_1 1_1 1_1 2_1 2_1 2_1 2_1 3_1 3_1 2_1 2_1 3_1 3_1\\
|f(1|2)|_{k=2} =&\ \ 1_2 2_2 1_2 2_2 2_2 3_2 2_2 3_2\\
|f(1|2)|_{k=3} =&\ \ 1_3 1_3 2_3 2_3\\
|f(1|2)|_{k=4} =&\ \ 1_4 2_4\\
|f(1|2)|_{k=5} =&\ \ 1_5
\end{aligned}
\tag{2.84}
$$

$$
\begin{aligned}
&(j=3, \quad period\ \ 2^7)\\
|f(1|3)|_{k=0} =&\ \ 1_0 2_0 1_0 2_0 1_0 2_0 1_0 2_0 2_0 2_0 2_0 2_0 2_0 2_0 2_0 2_0\\
&\ \ 1_0 2_0 1_0 2_0 1_0 2_0 1_0 2_0 2_0 2_0 2_0 2_0 2_0 2_0 2_0 2_0\\
&\ \ 1_0 2_0 1_0 2_0 1_0 2_0 1_0 2_0 2_0 2_0 2_0 2_0 2_0 2_0 2_0 2_0\\
&\ \ 1_0 2_0 1_0 2_0 1_0 2_0 1_0 2_0 2_0 2_0 2_0 2_0 2_0 2_0 2_0 2_0\\
&\ \ 2_0 2_0 2_0 2_0 2_0 2_0 2_0 2_0 2_0 1_0 2_0 1_0 2_0 1_0 2_0 1_0\\
&\ \ 2_0 2_0 2_0 2_0 2_0 2_0 2_0 2_0 2_0 1_0 2_0 1_0 2_0 1_0 2_0 1_0\\
&\ \ 2_0 2_0 2_0 2_0 2_0 2_0 2_0 2_0 2_0 1_0 2_0 1_0 2_0 1_0 2_0 1_0\\
&\ \ 2_0 2_0 2_0 2_0 2_0 2_0 2_0 2_0 2_0 1_0 2_0 1_0 2_0 1_0 2_0 1_0\\
|f(1|3)|_{k=1} =&\ \ 1_1 1_1 1_1 1_1 2_1 2_1 2_1 2_1 1_1 1_1 1_1 1_1 2_1 2_1 2_1 2_1\\
&\ \ 1_1 1_1 1_1 1_1 2_1 2_1 2_1 2_1 1_1 1_1 1_1 1_1 2_1 2_1 2_1 2_1\\
&\ \ 2_1 2_1 2_1 2_1 3_1 3_1 3_1 3_1 2_1 2_1 2_1 2_1 3_1 3_1 3_1 3_1\\
&\ \ 2_1 2_1 2_1 2_1 3_1 3_1 3_1 3_1 2_1 2_1 2_1 2_1 3_1 3_1 3_1 3_1\\
|f(1|3)|_{k=2} =&\ \ 1_2 1_2 2_2 2_2 1_2 1_2 2_2 2_2 1_2 1_2 2_2 2_2 1_2 1_2 2_2 2_2\\
&\ \ 2_2 2_2 3_2 3_2 2_2 2_2 3_2 3_2 2_2 2_2 3_2 3_2 2_2 2_2 3_2 3_2\\
|f(1|3)|_{k=3} =&\ \ 1_3 2_3 1_3 2_3 1_3 2_3 1_3 2_3 2_3 3_3 2_3 3_3 2_3 3_3 2_3 3_3\\
|f(1|3)|_{k=4} =&\ \ 1_4 1_4 1_4 1_4 2_4 2_4 2_4 2_4\\
|f(1|3)|_{k=5} =&\ \ 1_5 1_5 2_5 2_5\\
|f(1|3)|_{k=6} =&\ \ 1_6 2_6\\
|f(1|3)|_{k=7} =&\ \ 1_7
\end{aligned}
\tag{2.85}
$$

2 In the applications it would be enough to work only with the case of the maximal k for the line $|f(1|j)|$ because then the form of $|f(1|j)|$ is the simplest possible. For example, in (2.83)–(2.85) we see that for $|f(1|1)|, |f(1|2)|, |f(1|3)|$ the simplest forms are respectively the forms $1_3, 1_5, 1_7$ which take place respectively for the corresponding maximal values $k = 3, 5, 7$. Nevertheless, for the completeness of our presentation we write analytically the forms of lines $|f(1|j)|$ for all values of k.

Next we write again the examples (2.83)–(2.85) in a more compact manner.

$$(j=1, \quad \textit{period} \;\; 2^3)$$

$$\begin{aligned}
|f(1|1)|_{k=0} =&\ \ 1(1_0 2_0)1(2_0 2_0)1(2_0 2_0)1(2_0 1_0)\\
|f(1|1)|_{k=1} =&\ \ 1(1_1 2_1)1(2_1 3_1)\\
|f(1|1)|_{k=2} =&\ \ 1(1_2)1(2_2)\\
|f(1|1)|_{k=3} =&\ \ 1(1_3)
\end{aligned} \tag{2.86}$$

$$(j=2, \quad \textit{period} \;\; 2^5)$$

$$\begin{aligned}
|f(1|2)|_{k=0} =&\ \ 2[\,2(1_0 2_0)2(2_0 2_0)\,]2[\,2(2_0 2_0)2(2_0 1_0)\,]\\
|f(1|2)|_{k=1} =&\ \ 2[\,2(1_1)2(2_1)\,]2[\,2(2_1)2(3_1)\,]\\
|f(1|2)|_{k=2} =&\ \ 2[\,1(1_2)1(2_2)\,]2[\,1(2_2)1(3_2)\,]\\
|f(1|2)|_{k=3} =&\ \ 2(1_3)2(2_3)\\
|f(1|2)|_{k=4} =&\ \ 1(1_4)1(2_4)\\
|f(1|2)|_{k=5} =&\ \ 1(1_5)
\end{aligned} \tag{2.87}$$

$$(j=3, \quad \textit{period} \;\; 2^7)$$

$$\begin{aligned}
|f(1|3)|_{k=0} =&\ \ 4[\,4(1_0 2_0)4(2_0 2_0)\,]4[\,4(2_0 2_0)4(2_0 1_0)\,]\\
|f(1|3)|_{k=1} =&\ \ 4[\,4(1_1)4(2_1)\,]4[\,4(2_1)4(3_1)\,]\\
|f(1|3)|_{k=2} =&\ \ 4[\,2(1_2)2(2_2)\,]4[\,2(2_2)2(3_2)\,]\\
|f(1|3)|_{k=3} =&\ \ 4[\,1(1_3)1(2_3)\,]4[\,1(2_3)1(3_3)\,]\\
|f(1|3)|_{k=4} =&\ \ 4(1_4)4(2_4)\\
|f(1|3)|_{k=5} =&\ \ 2(1_5)2(2_5)\\
|f(1|3)|_{k=6} =&\ \ 1(1_6)1(2_6)\\
|f(1|3)|_{k=7} =&\ \ 1(1_7)
\end{aligned} \tag{2.88}$$

The notation in (2.86)–(2.88) is clear comparing each equation of (2.83)–(2.85) with its corresponding equation of (2.86)–(2.88). Thus it is clear that when, in (2.86)–(2.88), we write an integer before a string in parentheses or brackets we mean that this string is repeated the corresponding, to the integer, number of times: for example when we write $2(1_0 2_0)$ we mean $1_0 2_0 1_0 2_0$.

3 Next we write again (2.86)–(2.88) using the convention with the asterisks we have seen in (2.40),(2.43),(2.46), and (2.49), see also Note 2.2. So (2.89)–(2.91) are obtained.

$$
\begin{array}{rl}
 & (j = 1, \quad \textit{period} \;\; 2^3) \\
|f(1|1)|_{k=0} = & (1_0 2_0)(2_0 2_0)(2_0 2_0)(2_0 1_0) \\
|f(1|1)|_{k=1} = & (1_1 2_1)(2_1 3_1) \\
|f(1|1)|_{k=2} = & (1_2)(2_2) \\
|f(1|1)|_{k=3} = & (1_3)
\end{array}
\tag{2.89}
$$

$$
\begin{array}{rl}
 & (j = 2, \quad \textit{period} \;\; 2^5) \\
|f(1|2)|_{k=0} = & [\,(1_0 2_0 *)(2_0 2_0 *) *][\,(2_0 2_0 *)(2_0 1_0 *) *] \\
|f(1|2)|_{k=1} = & [\,(1_1 *)(2_1 *) *][\,(2_1 *)(3_1 *) *] \\
|f(1|2)|_{k=2} = & [\,(1_2)(2_2) *][\,(2_2)(3_2) *] \\
|f(1|2)|_{k=3} = & (1_3 *)(2_3 *) \\
|f(1|2)|_{k=4} = & (1_4)(2_4) \\
|f(1|2)|_{k=5} = & (1_5)
\end{array}
\tag{2.90}
$$

$$
\begin{array}{rl}
 & (j = 3, \quad \textit{period} \;\; 2^7) \\
|f(1|3)|_{k=0} = & [\,(1_0 2_0 **)(2_0 2_0 **) **][\,(2_0 2_0 **)(2_0 1_0 **) **] \\
|f(1|3)|_{k=1} = & [\,(1_1 **)(2_1 **) **][\,(2_1 **)(3_1 **) **] \\
|f(1|3)|_{k=2} = & [\,(1_2 *)(2_2 *) **][\,(2_2 *)(3_2 *) **] \\
|f(1|3)|_{k=3} = & [\,(1_3)(2_3) **][\,(2_3)(3_3) **] \\
|f(1|3)|_{k=4} = & (1_4 **)(2_4 **) \\
|f(1|3)|_{k=5} = & (1_5 *)(2_5 *) \\
|f(1|3)|_{k=6} = & (1_6)(2_6) \\
|f(1|3)|_{k=7} = & (1_7)
\end{array}
\tag{2.91}
$$

The meaning of the asterisks is clear in (2.89)–(2.91) if we compare these equations with their equivalent ones in (2.83)–(2.85) and in (2.86)–(2.88). For example by $1_0 2_0 **$ we mean $1_0 2_0 1_0 2_0 *$ or $1_0 2_0 1_0 2_0 1_0 2_0 1_0 2_0$.

4 Now having seen, in the examples (2.83)–(2.91), how the lines $|f(1|j)|$ for $j = 1, 2, 3$ are composed of the basic units $1_k, 2_k, 3_k$ we can see the same thing for the line $|f(1|j)|$ for the general value of j. This can be accomplished directly knowing, from Chapters 5,6, and 7 (especially Chapter 5) of the *Mathematical Diary 1993–1998* [helpful are also the examples in (2.38)–(2.40)], how is the form of the lines $|f(i|j)|$ for the general i's and j's. So in Table 2.9 we present how the line $|f(1|j)|$, for random j, is composed of the basic units $1_k, 2_k, 3_k$ for the various values of k.

Table 2.9 How the basic units $1_k, 2_k, 3_k$ form the line $|f(1|j)|$ for random j and for various values of k.

$$General\ formulae\ for\ line\ |f(1|j)| :$$

$$(k \in \{0, 1, 2, \cdots, 2j+1\}, \quad period\ \ 2^{2j+1}) \tag{2.92}$$

$$\begin{aligned}
|f(1|j)|_{k=0} &= 2^{j-1}[\,2^{j-1}(1_0 2_0)2^{j-1}(2_0 2_0)\,]2^{j-1}[\,2^{j-1}(2_0 2_0)2^{j-1}(2_0 1_0)\,] \\
|f(1|j)|_{1\leqslant k\leqslant j} &= 2^{j-1}[\,2^{j-k}(1_k)2^{j-k}(2_k)\,]2^{j-1}[\,2^{j-k}(2_k)2^{j-k}(3_k)\,] \\
|f(1|j)|_{j+1\leqslant k\leqslant 2j} &= [\,2^{2j-k}(1_k)2^{2j-k}(2_k)\,] \\
|f(1|j)|_{k=2j+1} &= (1_k)
\end{aligned} \tag{2.93}$$

or equivalently

$$
\begin{aligned}
|f(1|j)|_{k=0} &= [\,(1_0 2_0 (j-1)*)(2_0 2_0 (j-1)*)(j-1)*\,] \\
&\qquad\qquad [\,(2_0 2_0 (j-1)*)(2_0 1_0 (j-1)*)(j-1)*\,] \\
|f(1|j)|_{1\leqslant k\leqslant j} &= [\,(1_k (j-k)*)(2_k (j-k)*)(j-1)*\,] \\
&\qquad\qquad [\,(2_k (j-k)*)(3_k (j-k)*)(j-1)*\,] \\
|f(1|j)|_{j+1\leqslant k\leqslant 2j} &= (1_k (2j-k)*)(2_k (2j-k)*) \\
|f(1|j)|_{k=2j+1} &= (1_k)
\end{aligned}
\tag{2.94}
$$

In (2.93), as in (2.86)–(2.88), an integer before a string means repetition of the string for a number of times equal to this integer.

Everywhere in (2.94) we have the notation $\alpha(n)*$ with α a string and n an integer, for example, $1_0 2_0 (j-1)*$ where $\alpha = 1_0 2_0$ and $n = j-1$. This means that at the right side of string α there are n asterisks $*$ whose meaning is clear from (2.89)–(2.91). So it is

$$2^n \alpha \equiv \underbrace{\alpha\,\alpha\cdots\alpha}_{2^n \text{ letters } \alpha} = \alpha \underbrace{*\,*\cdots*}_{n \text{ asterisks } *} \equiv \alpha(n)* \tag{2.95}$$

Inside (2.95) we have

$$2^n \alpha \equiv \alpha(n)* \tag{2.96}$$

with the help of which (2.93) is transformed directly into (2.94). Thus in the example of $\alpha(n)*$ with $\alpha = 1_0 2_0$ and $n = j-1$ if we put $j = 3$, hence $n = 2$, we obtain

$$1_0 2_0 (2)* = 1_0 2_0 ** = 1_0 2_0 1_0 2_0 * = 1_0 2_0 1_0 2_0 1_0 2_0 1_0 2_0 \tag{2.97}$$

---

5 Next we consider the example of $|f(1|4)|$ for checking the use of the general formulae of Table 2.9.

Knowing the general form of line $|f(i|j)|$ [see Chapters 5,6,7 of the *Mathematical Diary 1993–1998* but also the examples (2.38)–(2.40) and (2.83)–(2.85) and (2.86)–(2.88)] we can see easily that

$$|f(1|4)| = 8[\,8(01)8(11)\,]8[\,8(11)8(10)\,] \tag{2.98}$$

with period 2^9 [we have used the notation of (2.86)–(2.88) and (2.93) regarding the numerical factors]. Thus we can obtain easily, as we did previously for the examples of $|f(1|j)|$ for $j = 1, 2, 3$ in (2.86)–(2.88), that

$$
(j = 4, \quad period \;\; 2^9)
$$

$$
\begin{array}{ll}
|f(1|4)|_{k=0} = & 8[\,8(1_0 2_0)8(2_0 2_0)\,]8[\,8(2_0 2_0)8(2_0 1_0)\,] \\
|f(1|4)|_{k=1} = & 8[\,8(1_1)8(2_1)\,]8[\,8(2_1)8(3_1)\,] \\
|f(1|4)|_{k=2} = & 8[\,4(1_2)4(2_2)\,]8[\,4(2_2)4(3_2)\,] \\
|f(1|4)|_{k=3} = & 8[\,2(1_3)2(2_3)\,]8[\,2(2_3)2(3_3)\,] \\
|f(1|4)|_{k=4} = & 8[\,1(1_4)1(2_4)\,]8[\,1(2_4)1(3_4)\,] \\
|f(1|4)|_{k=5} = & 8(1_5)8(2_5) \\
|f(1|4)|_{k=6} = & 4(1_6)4(2_6) \\
|f(1|4)|_{k=7} = & 2(1_7)2(2_7) \\
|f(1|4)|_{k=8} = & 1(1_8)1(2_8) \\
|f(1|4)|_{k=9} = & 1(1_9)
\end{array}
\qquad (2.99)
$$

Again the notation in (2.99) is as in (2.86)–(2.88) and in (2.93).

As for the purpose of the example (2.99), i.e. to check the general formulae of Table 2.9, we can see indeed that (2.99) is produced directly from (2.93) if we put $j = 4$.

2.7 Generalizing for $|f(i|j)|$ — Determining by recursive algorithm the basic units of which the lines $|f(i|j)|$ [i.e. the lines $2P^{\,i}_{\,j}$] are composed.

1 Now we are going to do for the general case of line $|f(i|j)|$ what we have done in Section 2.4 for the special case of $|f(1|j)|$. That is, we are going to find, by recursive algorithm, the basic units of which the line $|f(i|j)|$ is composed.

We work as in Section 2.4 but we present the results briefly since we have now the previous experience. First we consider cases $|f(2|j)|$ and $|f(3|j)|$ as examples. Practically we repeat the procedure of (2.50)–(2.52) avoiding unnecessary (i.e. clear, well-understood) steps. Having also in mind the analytical presentation of the first few lines $|f(i|j)|$ in (2.38)–(2.40) may be of help.

Let us consider $|f(2|j)|$ for $j = 1, 2, 3$ and write the basic units for the various values of k.

(We use again convention (2.67) for the strings l, that is, $l \in \{0, 1, 2, \cdots, 2^{2^k} - 1\}$)

$$|f(2|1)|, \quad \textit{period} \;\; 2^4$$

$$\textit{the units}$$

$$\begin{array}{lcl} \text{for } \; k=0: & 0,1 & \equiv 1_0, 2_0 \\ \text{for } \; k=1: & 00,11 & \equiv 1_1, 2_1 \\ \text{for } \; k=2: & 0011, 1111, 1100 & \equiv 1_2, 2_2, 3_2 \\ \text{for } \; k=3: & 00111111, 11111100 & \equiv 1_3, 2_3 \\ \text{for } \; k=4: & 0011111111111100 & \equiv 1_4 \end{array} \tag{2.100}$$

$$|f(2|2)|, \quad \textit{period} \;\; 2^6$$

$$\textit{the units}$$

$$\begin{array}{lcl} \text{for } \; k=0: & 0,1 & \equiv 1_0, 2_0 \\ \text{for } \; k=1: & 00,11 & \equiv 1_1, 2_1 \\ \text{for } \; k=2: & 0011, 1111, 1100 & \equiv 1_2, 2_2, 3_2 \\ \text{for } \; k=3: & 00110011, 11111111, 11001100 & \equiv 1_3, 2_3, 3_3 \\ \text{for } \; k=4: & 0011001111111111, 1111111111001100 & \equiv 1_4, 2_4 \\ \text{for } \; k=5: & 00110011111111110011001111111111, & \\ & 11111111110011001111111111001100 & \equiv 1_5, 2_5 \\ \text{for } \; k=6: & 00110011111111110011001111111111 & \\ & 11111111110011001111111111001100 & \equiv 1_6 \end{array} \tag{2.101}$$

$$|f(2|3)|, \quad \textit{period} \;\; 2^8$$

$$\textit{the units}$$

$$\begin{array}{lcl}
\text{for } k=0: & 0,1 & \equiv 1_0, 2_0 \\
\text{for } k=1: & 00,11 & \equiv 1_1, 2_1 \\
\text{for } k=2: & 0011,1111,1100 & \equiv 1_2, 2_2, 3_2 \\
\text{for } k=3: & 00110011,11111111,11001100 & \equiv 1_3, 2_3, 3_3 \\
\text{for } k=4: & 0011001100110011, & \\
 & 1111111111111111, & \\
 & 1100110011001100 & \equiv 1_4, 2_4, 3_4 \\
\text{for } k=5: & 00110011001100111111111111111111, & \\
 & 11111111111111111100110011001100 & \equiv 1_5, 2_5 \\
\text{for } k=6: & 2(1_5), 2(2_5) & \equiv 1_6, 2_6 \\
\text{for } k=7: & 2(1_6), 2(2_6) & \equiv 1_7, 2_7 \\
\text{for } k=8: & 1_7 2_7 & \equiv 1_8
\end{array} \tag{2.102}$$

2 Similarly we consider the example of $|f(3|j)|$.

$$|f(3|1)|, \quad \textit{period } 2^5$$

$$\textit{the units}$$

$$\begin{array}{lcl}
\text{for } k=0: & 0,1 & \equiv 1_0, 2_0 \\
\text{for } k=1: & 00,11 & \equiv 1_1, 2_1 \\
\text{for } k=2: & 0000,1111 & \equiv 1_2, 2_2 \\
\text{for } k=3: & 1_2 2_2, 2_2 2_2, 2_2 1_2 & \equiv 1_3, 2_3, 3_3 \\
\text{for } k=4: & 1_3 2_3, 2_3 3_3 & \equiv 1_4, 2_4 \\
\text{for } k=5: & 1_4 2_4 & \equiv 1_5
\end{array} \tag{2.103}$$

$$|f(3|2)|, \quad \textit{period } 2^7$$

$$\textit{the units}$$

$$\begin{array}{lcl}
\text{for } k=0: & 0,1 & \equiv 1_0, 2_0 \\
\text{for } k=1: & 00,11 = 1_0 1_0, 2_0 2_0 & \equiv 1_1, 2_1 \\
\text{for } k=2: & 0000,1111 = 1_1 1_1, 2_1 2_1 & \equiv 1_2, 2_2 \\
\text{for } k=3: & 1_2 2_2, 2_2 2_2, 2_2 1_2 & \equiv 1_3, 2_3, 3_3 \\
\text{for } k=4: & 1_3 1_3, 2_3 2_3, 3_3 3_3 & \equiv 1_4, 2_4, 3_4 \\
\text{for } k=5: & 1_4 2_4, 2_4 3_4 & \equiv 1_5, 2_5 \\
\text{for } k=6: & 1_5 1_5, 2_5 2_5 & \equiv 1_6, 2_6 \\
\text{for } k=7: & 1_6 2_6 & \equiv 1_7
\end{array} \tag{2.104}$$

$$|f(3|3)|, \quad \textit{period } 2^9$$

$$\begin{array}{llll} & \textit{the units} & & \\ \text{for } k=0: & 0,1 & \equiv 1_0,2_0 & \\ \text{for } k=1: & 00,11 = 1_01_0, 2_02_0 & \equiv 1_1,2_1 & \\ \text{for } k=2: & 0000,1111 = 1_11_1, 2_12_1 & \equiv 1_2,2_2 & \\ \text{for } k=3: & 1_22_2, 2_22_2, 2_21_2 & \equiv 1_3,2_3,3_3 & \\ \text{for } k=4: & 1_31_3, 2_32_3, 3_33_3 & \equiv 1_4,2_4,3_4 & \\ \text{for } k=5: & 1_41_4, 2_42_4, 3_43_4 & \equiv 1_5,2_5,3_5 & \\ \text{for } k=6: & 1_52_5, 2_53_5 & \equiv 1_6,2_6 & \\ \text{for } k=7: & 1_61_6, 2_62_6 & \equiv 1_7,2_7 & \\ \text{for } k=8: & 1_71_7, 2_72_7 & \equiv 1_8,2_8 & \\ \text{for } k=9: & 1_82_8 & \equiv 1_9 & \end{array} \tag{2.105}$$

3 Next we wish to construct the corresponding to Table 2.5 for the general case of the line $|f(i|j)|$ (production of the basic units with the help of *recursive formulae*). This can be easily done taking into account the above examples. We use again for the strings the convention (2.67), that is, $l \in \{0, 1, 2, \cdots, 2^{2^k} - 1\}$. So Table 2.10 results.

Table 2.10 Recursive formulae by which the basic units, that compose the lines $|f(i|j)|$, are determined for the various cases k [the generalization of Table 2.5 for lines $|f(i|j)|$].

$$Units\ for\ |f(i|j)|:$$

$$The\ entities\ k, \alpha, \beta:$$

$$k \in \{0, 1, 2, \cdots, 2j + i\} \tag{2.106}$$

For α and β, each of which takes values from the set $\{1, 2, 3\}$, and for any k it is

$$\alpha_{k-1}\beta_{k-1} \equiv \alpha_{k-1} \cdot 2^{2^{k-1}} + \beta_{k-1} \in \{0, 1, 2, \cdots, 2^{2^k} - 1\} \tag{2.107}$$

$$The\ units:$$

$$k=0: \quad \underset{1_k}{0}, \quad \underset{2_k}{1} \quad \in \{0,1,2,\cdots,2^{2^k}-1\} \tag{2.108a}$$

(for $i \geqslant 2$)

$$1 \leqslant k \leqslant i-1: \quad \underset{1_k}{1_{k-1}1_{k-1}}, \quad \underset{2_k}{2_{k-1}2_{k-1}} \tag{2.108b}$$

$$k=i: \quad \underset{1_k}{1_{k-1}2_{k-1}}, \quad \underset{2_k}{2_{k-1}2_{k-1}}, \quad \underset{3_k}{2_{k-1}1_{k-1}} \tag{2.108c}$$

(for $j \geqslant 2$)

$$i+1 \leqslant k \leqslant i+j-1: \quad \underset{1_k}{1_{k-1}1_{k-1}}, \quad \underset{2_k}{2_{k-1}2_{k-1}}, \quad \underset{3_k}{3_{k-1}3_{k-1}} \tag{2.108d}$$

$$k=i+j: \quad \underset{1_k}{1_{k-1}2_{k-1}}, \quad \underset{2_k}{2_{k-1}3_{k-1}} \tag{2.108e}$$

(for $j \geqslant 2$)

$$i+j+1 \leqslant k \leqslant i+2j-1: \quad \underset{1_k}{1_{k-1}1_{k-1}}, \quad \underset{2_k}{2_{k-1}2_{k-1}} \tag{2.108f}$$

$$k=i+2j: \quad \underset{1_k}{1_{k-1}2_{k-1}} \tag{2.108g}$$

2.8 Expressing the units of the algorithm of Table 2.10, which is for the general $|f(i|j)|$, in a nonrecursive manner.

1 For the special case of $|f(1|j)|$ from Table 2.5 we obtain the equivalent Table 2.6 where the units are expressed in nonrecursive form. The same thing we are doing now for the general $|f(i|j)|$ and from Table 2.10 we obtain the equivalent Table 2.11 where the units appear in nonrecursive form. Table 2.11 is produced through a direct, familiar procedure so further descriptions are not necessary.

Table 2.11 A table equivalent to Table 2.10 but with the basic units expressed in nonrecursive form [the generalization of Table 2.6 for lines $|f(i|j)|$].

$$Units\ for\ |f(i|j)|\ (nonrecursively):$$

$$the\ values\ of\ k:$$

$$k \in \{0, 1, 2, \cdots, 2j+i\} \tag{2.109}$$

$$the\ units:$$

$$\begin{array}{llll} k=0: & 0 & \equiv 1_0 & = 1_k \\ & 1 & \equiv 2_0 & = 2_k \end{array} \tag{2.110a}$$

$$\begin{array}{lcr} \text{(for } i \geqslant 2) & & \\ 1 \leqslant k \leqslant i-1: & 0 & \equiv 1_k \\ & (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^2}+1)\cdots(2^{2^{k-1}}+1) & \equiv 2_k \end{array} \tag{2.110b}$$

$$k=i: \quad 1_{k-1}2_{k-1} = 1_{k-1} \cdot 2^{2^{k-1}} + 2_{k-1} =$$

$$(2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^2}+1)\cdots(2^{2^{i-2}}+1)\cdot 1 \qquad \equiv 1_k$$

$$2_{k-1}2_{k-1} = 2_{k-1} \cdot 2^{2^{k-1}} + 2_{k-1} =$$

$$(2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^2}+1)\cdots(2^{2^{i-2}}+1)$$

$$\cdot(2^{2^{i-1}}+1) \qquad \equiv 2_k$$

$$2_{k-1}1_{k-1} = 2_{k-1} \cdot 2^{2^{k-1}} + 1_{k-1} =$$

$$(2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^2}+1)\cdots(2^{2^{i-2}}+1)\cdot 2^{2^{i-1}} \quad \equiv 3_k \tag{2.110c}$$

$$
\begin{aligned}
&(\text{for } \ j \geqslant 2) \\
&i+1 \leqslant k \leqslant i+j-1: \quad 1_{k-1}1_{k-1} = \\
&1_{k-1} \cdot 2^{2^{k-1}} + 1_{k-1} = 1_{k-1} \cdot (2^{2^{k-1}} + 1) = \\
&(2^{2^0}+1) \cdot (2^{2^1}+1) \cdot (2^{2^2}+1) \cdots (2^{2^{i-2}}+1) \\
&\cdot 1 \cdot (2^{2^i}+1) \cdot (2^{2^{i+1}}+1) \cdots (2^{2^{k-1}}+1) \equiv 1_k \\
&2_{k-1}2_{k-1} = \\
&2_{k-1} \cdot 2^{2^{k-1}} + 2_{k-1} = 2_{k-1} \cdot (2^{2^{k-1}} + 1) = \\
&(2^{2^0}+1) \cdot (2^{2^1}+1) \cdot (2^{2^2}+1) \\
&\cdots (2^{2^{i-2}}+1) \cdot (2^{2^{i-1}}+1) \cdot \\
&(2^{2^i}+1) \cdot (2^{2^{i+1}}+1) \cdots (2^{2^{k-1}}+1) \equiv 2_k \\
&3_{k-1}3_{k-1} = \\
&3_{k-1} \cdot 2^{2^{k-1}} + 3_{k-1} = 3_{k-1} \cdot (2^{2^{k-1}} + 1) = \\
&(2^{2^0}+1) \cdot (2^{2^1}+1) \cdot (2^{2^2}+1) \\
&\cdots (2^{2^{i-2}}+1) \cdot 2^{2^{i-1}} \cdot (2^{2^i}+1) \\
&\cdot (2^{2^{i+1}}+1) \cdots (2^{2^{k-1}}+1) \equiv 3_k
\end{aligned} \tag{2.110d}
$$

$k = i + j :\quad 1_{k-1}2_{k-1} = 1_{k-1} \cdot 2^{2^{k-1}} + 2_{k-1} =$

$$(2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^2}+1)\cdots(2^{2^{i-2}}+1)$$

$$\underbrace{\cdot(2^{2^i}+1)\cdot(2^{2^{i+1}}+1)\cdots(2^{2^{i+j-2}}+1)}_{\text{for } j=1 \text{ it does NOT exist!}}$$

$$\cdot[\,2^{2^{i+j-1}}+2^{2^{i-1}}+1\,] \equiv 1_k$$

$$2_{k-1}3_{k-1} = 2_{k-1}\cdot 2^{2^{k-1}} + 3_{k-1} =$$

$$(2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^2}+1)\cdots(2^{2^{i-2}}+1)$$

$$\underbrace{\cdot(2^{2^i}+1)\cdot(2^{2^{i+1}}+1)\cdots(2^{2^{i+j-2}}+1)}_{\text{for } j=1 \text{ it does NOT exist!}}$$

$$\cdot[\,2^{2^{i+j-1}+2^{i-1}}+2^{2^{i+j-1}}+2^{2^{i-1}}\,] \equiv 2_k \tag{2.110e}$$

(for $j \geqslant 2$)
$i+j+1 \leqslant k \leqslant i+2j-1 :$

$$1_{k-1}1_{k-1} =$$

$$1_{k-1}\cdot 2^{2^{k-1}} + 1_{k-1} = 1_{k-1}\cdot(2^{2^{k-1}}+1) =$$

$$(2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^2}+1)\cdots(2^{2^{i-2}}+1)$$

$$\cdot(2^{2^i}+1)\cdot(2^{2^{i+1}}+1)\cdots(2^{2^{i+j-2}}+1)$$

$$\cdot[\,2^{2^{i+j-1}}+2^{2^{i-1}}+1\,]\cdot$$

$$(2^{2^{i+j}}+1)\cdot(2^{2^{i+j+1}}+1)\cdots(2^{2^{k-1}}+1) \equiv 1_k$$

$$
\begin{aligned}
&2_{k-1}2_{k-1} = \\
&2_{k-1} \cdot 2^{2^{k-1}} + 2_{k-1} = 2_{k-1} \cdot (2^{2^{k-1}} + 1) = \\
&(2^{2^0} + 1) \cdot (2^{2^1} + 1) \cdot (2^{2^2} + 1) \cdots (2^{2^{i-2}} + 1) \\
&\cdot (2^{2^i} + 1) \cdot (2^{2^{i+1}} + 1) \cdots (2^{2^{i+j-2}} + 1) \\
&\cdot [\, 2^{2^{i+j-1}+2^{i-1}} + 2^{2^{i+j-1}} + 2^{2^{i-1}} \,] \cdot \\
&(2^{2^{i+j}} + 1) \cdot (2^{2^{i+j+1}} + 1) \cdots (2^{2^{k-1}} + 1) \equiv 2_k
\end{aligned} \tag{2.110f}
$$

$$
\begin{aligned}
k = i + 2j : \quad &1_{k-1}2_{k-1} = 1_{k-1} \cdot 2^{2^{k-1}} + 2_{k-1} = \\
&(2^{2^0} + 1) \cdot (2^{2^1} + 1) \cdot (2^{2^2} + 1) \cdots (2^{2^{i-2}} + 1) \\
&\underbrace{\cdot (2^{2^i} + 1) \cdot (2^{2^{i+1}} + 1) \cdots (2^{2^{i+j-2}} + 1)}_{\text{for } j\text{=1 it does NOT exist!}} \\
&\cdot [\, 2^{2^{i+2j-1}+2^{i+j-1}} + 2^{2^{i+2j-1}+2^{i-1}} \\
&+ 2^{2^{i+j-1}+2^{i-1}} + 2^{2^{i+2j-1}} + 2^{2^{i+j-1}} + 2^{2^{i-1}} \,] \\
&\underbrace{\cdot (2^{2^{i+j}} + 1) \cdot (2^{2^{i+j+1}} + 1) \cdots (2^{2^{i+2j-2}} + 1)}_{\text{for } j\text{=1 it does NOT exist!}} \equiv 1_k
\end{aligned} \tag{2.110g}
$$

Note 2.12 ATTENTION! In the previous pages, and especially from Sec. 2.4,Subsec. 5 on, in some places in the general formulae there seems to be a problem concerning the indices k. In those places, as we can see in the examples, the problem really *does not exist!* We just follow, for indices k, the most natural interpretation.

For example, if $j = 1$ and somewhere we have for the values of k that $j+2 \leqslant k \leqslant 2j$, i.e. that $3 \leqslant k \leqslant 2$, we just ignore this case and we proceed directly to the value $k = 2j + 1 = 3$ from the previous value $k = j + 1 = 2$; also if [see (2.110e) and (2.110g)] we have the product $(2^{2^i}+1)\cdot(2^{2^{i+1}}+1)\cdots(2^{2^{i+j-2}}+1)$ then for $j = 1$, when it is $(2^{2^i}+1)\cdot(2^{2^{i+1}}+1)\cdots(2^{2^{i-1}}+1)$ i.e. an unnatural sequence, we just ignore the specific product considering it as 1 [we emphasize this in (2.110e) and (2.110g) using the "underbraces" with the phrase "it does not exist" to denote that in this unnatural case we just put 1]. So where such problematic situations appear, regarding the values of k and i and j, we interpret them in the clear, from the context and the procedure (also the examples are of help), way and no confusion takes place! □

---

2 Essentially the results obtained so far, which in compact form appear in the tables, are produced through inductive work based on examples. In the opposite direction these tables, when completely constructed, may be clarified through examples. At this point, before proceeding next, we are going to present $|f(2|3)|$ and $|f(3|2)|$ as examples for Tables 2.10 and 2.11 checking that way the tables.

First we do the example $|f(2|3)|$.

We start by writing the units of $|f(2|3)|$ obtained from (2.102) [see also Sec. 2.3]:

$$\begin{array}{lc} \text{for } k = 0: & 0, 1 \\ \text{for } k = 1: & 0, 3 \\ \text{for } k = 2: & 3, 15, 12 \\ \text{for } k = 3: & 51, 255, 204 \\ \vdots & \vdots \end{array} \tag{2.111}$$

[The units in (2.111), for $k > 0$, are as in Sec. 2.3 minus one because for those k's in Sec. 2.3 we use convention (2.28) for the strings whereas in (2.111) we use convention (2.29) or (2.67).]

Next we write the units of $|f(2|3)|$ obtained from the algorithm of Table 2.10:

$$
\begin{array}{ll}
\text{for } k=0: & 0,1 \\
\text{for } k=1: & 00,11=(0\cdot 2+0),(1\cdot 2+1)= \\
 & 0,3 \\
\text{for } k=2: & 03,33,30=(0\cdot 4+3),(3\cdot 4+3),(3\cdot 4+0)= \\
 & 3,15,12 \\
\text{for } k=3: & 3\;3,15\;15,12\;12= \\
 & (3\cdot 16+3),(15\cdot 16+15),(12\cdot 16+12)= \\
 & 51,255,204 \\
\text{for } k=4: & 51\;51,255\;255,204\;204= \\
 & (51\cdot 2^8+51),(255\cdot 2^8+255),(204\cdot 2^8+204) \\
 & \equiv 1_4,2_4,3_4 \\
\text{for } k=5: & 1_4\;2_4,2_4\;3_4= \\
 & (1_4\cdot 2^{16}+2_4),(2_4\cdot 2^{16}+3_4) \\
 & \equiv 1_5,2_5 \\
\text{for } k=6: & 1_5\;1_5,2_5\;2_5= \\
 & (1_5\cdot 2^{32}+1_5),(2_5\cdot 2^{32}+2_5) \\
 & =1_5\cdot(2^{32}+1),2_5\cdot(2^{32}+1) \\
 & \equiv 1_6,2_6 \\
\text{for } k=7: & 1_6\;1_6,2_6\;2_6= \\
 & (1_6\cdot 2^{64}+1_6),(2_6\cdot 2^{64}+2_6) \\
 & =1_6\cdot(2^{64}+1),2_6\cdot(2^{64}+1) \\
 & =1_5\cdot(2^{32}+1)\cdot(2^{64}+1),2_5\cdot(2^{32}+1)\cdot(2^{64}+1) \\
 & \equiv 1_7,2_7 \\
\text{for } k=8: & 1_7\;2_7= \\
 & (1_7\cdot 2^{128}+2_7) \\
 & =(2^{64}+1)\cdot(1_6\cdot 2^{128}+2_6) \\
 & =(2^{32}+1)\cdot(2^{64}+1)\cdot(1_5\cdot 2^{128}+2_5) \\
 & \equiv 1_8
\end{array}
\tag{2.112}
$$

Next we write the units of $|f(2|3)|$ obtained from the algorithm of Table 2.11:

$$
\begin{array}{ll}
\text{for } \; k=0: & 0,1 \\
\text{for } \; k=1: & 0,(2^{2^0}+1)=0,3 \\
\text{for } \; k=2: & (2^{2^0}+1),(2^{2^0}+1)\cdot(2^{2^1}+1),(2^{2^0}+1)\cdot 2^{2^1} \\
 & =3,15,12 \\
\text{for } \; k=3: & (2^{2^0}+1)\cdot 1\cdot(2^{2^2}+1), \\
 & (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^2}+1), \\
 & (2^{2^0}+1)\cdot 2^{2^1}\cdot(2^{2^2}+1) \\
 & =51,255,204 \\
\text{for } \; k=4: & (2^{2^0}+1)\cdot 1\cdot(2^{2^2}+1)\cdot(2^{2^3}+1), \\
 & (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^2}+1)\cdot(2^{2^3}+1), \\
 & (2^{2^0}+1)\cdot 2^{2^1}\cdot(2^{2^2}+1)\cdot(2^{2^3}+1) \\
 & \equiv 1_4,2_4,3_4 \\
\text{for } \; k=5: & (2^{2^0}+1)\cdot(2^{2^2}+1)\cdot(2^{2^3}+1) \\
 & \cdot[\,2^{2^4}+2^{2^1}+1\,], \\
 & (2^{2^0}+1)\cdot(2^{2^2}+1)\cdot(2^{2^3}+1) \\
 & \cdot[\,2^{2^4+2^1}+2^{2^4}+2^{2^1}\,] \\
 & \equiv 1_5,2_5 \\
\text{for } \; k=6: & (2^{2^0}+1)\cdot(2^{2^2}+1)\cdot(2^{2^3}+1) \\
 & \cdot[\,2^{2^4}+2^{2^1}+1\,]\cdot(2^{2^5}+1), \\
 & (2^{2^0}+1)\cdot(2^{2^2}+1)\cdot(2^{2^3}+1) \\
 & \cdot[\,2^{2^4+2^1}+2^{2^4}+2^{2^1}\,]\cdot(2^{2^5}+1) \\
 & \equiv 1_6,2_6 \\
\text{for } \; k=7: & (2^{2^0}+1)\cdot(2^{2^2}+1)\cdot(2^{2^3}+1) \\
 & \cdot[\,2^{2^4}+2^{2^1}+1\,]\cdot(2^{2^5}+1)\cdot(2^{2^6}+1), \\
 & (2^{2^0}+1)\cdot(2^{2^2}+1)\cdot(2^{2^3}+1) \\
 & \cdot[\,2^{2^4+2^1}+2^{2^4}+2^{2^1}\,]\cdot(2^{2^5}+1)\cdot(2^{2^6}+1) \\
 & \equiv 1_7,2_7 \\
\text{for } \; k=8: & (2^{2^0}+1)\cdot(2^{2^2}+1)\cdot(2^{2^3}+1) \\
 & \cdot[\,2^{2^7+2^4}+2^{2^7+2^1}+2^{2^4+2^1}+2^{2^7}+2^{2^4}+2^{2^1}\,] \\
 & \cdot(2^{2^5}+1)\cdot(2^{2^6}+1) \\
 & \equiv 1_8
\end{array}
\tag{2.113}
$$

Comparing the results in (2.111), (2.112), and (2.113) for the various values of k we see that they are same. Especially comparing (2.112) and (2.113) we can see directly that for $k = 0, 1, 2, 3$ the results are same. As for the values $k = 4, 5, 6, 7, 8$ we can see, without special difficulty, that when the results are same in (2.112) and (2.113) for

k then the results in (2.112) and (2.113) are same also for $k+1$. Thus the results in (2.112) and (2.113) are same for all the values of k.

Next we do the example $|f(3|2)|$ in a similar manner.

We start by writing the units of $|f(3|2)|$ obtained from (2.104) [see also Sec. 2.3]:

$$\begin{array}{ll} \text{for } k=0: & 0,1 \\ \text{for } k=1: & 0,3 \\ \text{for } k=2: & 0,15 \\ \text{for } k=3: & 15,255,240 \\ \vdots & \vdots \end{array} \tag{2.114}$$

[In (2.114) we have the same style for the presentation of the units as in (2.111).]

Next we write the units of $|f(3|2)|$ obtained from the algorithm of Table 2.10:

$$\begin{array}{ll} \text{for } k=0: & 0,1 \\ \text{for } k=1: & 00,11=(0\cdot 2+0),(1\cdot 2+1)= \\ & 0,3 \\ \text{for } k=2: & 00,33=(0\cdot 4+0),(3\cdot 4+3)= \\ & 0,15 \\ \text{for } k=3: & 0\ 15,15\ 15,15\ 0= \\ & (0\cdot 16+15),(15\cdot 16+15),(15\cdot 16+0)= \\ & 15,255,240 \\ \text{for } k=4: & 15\ 15,255\ 255,240\ 240= \\ & (15\cdot 2^8+15),(255\cdot 2^8+255),(240\cdot 2^8+240) \\ & =15\cdot(2^8+1),255\cdot(2^8+1),240\cdot(2^8+1) \\ & \equiv 1_4,2_4,3_4 \\ \text{for } k=5: & 1_4\ 2_4,2_4\ 3_4= \\ & (1_4\cdot 2^{16}+2_4),(2_4\cdot 2^{16}+3_4) \\ & \equiv 1_5,2_5 \end{array}$$

$$
\begin{array}{ll}
\text{for } \ k=6: & 1_5\ 1_5, 2_5\ 2_5 = \\
& (1_5 \cdot 2^{32} + 1_5), (2_5 \cdot 2^{32} + 2_5) \\
& = 1_5 \cdot (2^{32}+1), 2_5 \cdot (2^{32}+1) \\
& \equiv 1_6, 2_6 \\
\text{for } \ k=7: & 1_6\ 2_6 = \\
& (1_6 \cdot 2^{64} + 2_6) \\
& \equiv 1_7
\end{array}
\tag{2.115}
$$

Next we write the units of $|f(3|2)|$ obtained from the algorithm of Table 2.11:

$$
\begin{array}{ll}
\text{for } \ k=0: & 0, 1 \\
\text{for } \ k=1: & 0, (2^{2^0}+1) = 0, 3 \\
\text{for } \ k=2: & 0, (2^{2^0}+1)\cdot(2^{2^1}+1) = 0, 15 \\
\text{for } \ k=3: & (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot 1, \\
& (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^2}+1), \\
& (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot 2^{2^2} \\
& = 15, 255, 240 \\
\text{for } \ k=4: & (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot 1\cdot(2^{2^3}+1), \\
& (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^2}+1)\cdot(2^{2^3}+1), \\
& (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot 2^{2^2}\cdot(2^{2^3}+1) \\
& \equiv 1_4, 2_4, 3_4 \\
\text{for } \ k=5: & (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^3}+1) \\
& \cdot[\,2^{2^4}+2^{2^2}+1\,], \\
& (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^3}+1) \\
& \cdot[\,2^{2^4+2^2}+2^{2^4}+2^{2^2}\,] \\
& \equiv 1_5, 2_5 \\
\text{for } \ k=6: & (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^3}+1) \\
& \cdot[\,2^{2^4}+2^{2^2}+1\,]\cdot(2^{2^5}+1), \\
& (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^3}+1) \\
& \cdot[\,2^{2^4+2^2}+2^{2^4}+2^{2^2}\,]\cdot(2^{2^5}+1) \\
& \equiv 1_6, 2_6
\end{array}
$$

$$\text{for } k=7: \quad (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^3}+1)$$
$$\cdot[\,2^{2^6+2^4}+2^{2^6+2^2}+2^{2^4+2^2}+2^{2^6}+2^{2^4}+2^{2^2}\,]\cdot(2^{2^5}+1)$$
$$\equiv 1_7 \tag{2.116}$$

We can see directly that the results in (2.114),(2.115), and (2.116) are same for $k = 0, 1, 2, 3$. Also we can see easily that the results in (2.115) and (2.116) are same for $k = 4, 5, 6, 7$. Thus the results in (2.115) and (2.116) are same for all values of k.

---

3 Now having checked Tables 2.10 and 2.11 with the help of the examples $|f(2|3)|$ and $|f(3|2)|$ we come back to our main work of the present section (Sec. 2.8) and we write Table 2.11 more compactly thus producing Table 2.13 where we use again the style of Table 2.7. In fact Table 2.13 is for Table 2.11 what Table 2.7 is for Table 2.6 and Tables 2.6 and 2.7 are respectively just Tables 2.11 and 2.13 for $i = 1$.

Next we present Table 2.13 directly since everything is clear. (In the following the content of Note 2.12 is always relevant.)

Table 2.13 A more compact form of Table 2.11. (The generalized version of Table 2.7 for random i, i.e., for $|f(i|j)|$).

$$Units\ for\ |f(i|j)|\ (nonrecursively):$$

$$the\ values\ of\ k:$$

$$k \in \{0, 1, 2, \cdots, 2j+i\} \tag{2.117}$$

$$various\ entities:$$

$$A_m \equiv (2^{2^m} + 1) \tag{2.118}$$

$$\begin{aligned}
X &\equiv [\, 2^{2^{i+j-1}} + 2^{2^{i-1}} + 1 \,] \\
Y &\equiv [\, 2^{2^{i+j-1}+2^{i-1}} + 2^{2^{i+j-1}} + 2^{2^{i-1}} \,] \\
Z &\equiv [\, 2^{2^{i+2j-1}+2^{i+j-1}} + 2^{2^{i+2j-1}+2^{i-1}} + 2^{2^{i+j-1}+2^{i-1}} \\
&\qquad +2^{2^{i+2j-1}} + 2^{2^{i+j-1}} + 2^{2^{i-1}} \,]
\end{aligned} \tag{2.119}$$

$$\begin{aligned}
(1_0, 2_0) &\equiv (0, 1) \\
(a, b, c) &\equiv (1, 2^{2^{i-1}} + 1, 2^{2^{i-1}})
\end{aligned} \tag{2.120}$$

$$the\ units:$$

$$k = 0: \qquad (1_k, 2_k) = (1_0, 2_0) \tag{2.121a}$$

$$\begin{aligned}
&(\text{for } \ i \geqslant 2) \\
&1 \leqslant k \leqslant i-1: \qquad 1_k \equiv 0 \\
&\qquad\qquad\qquad\qquad 2_k \equiv A_0 \cdot A_1 \cdot A_2 \cdots A_{k-1}
\end{aligned} \tag{2.121b}$$

$$\begin{aligned}
k = i: \qquad 1_k &\equiv A_0 \cdot A_1 \cdot A_2 \cdots A_{i-2} \cdot a \\
2_k &\equiv A_0 \cdot A_1 \cdot A_2 \cdots A_{i-2} \cdot b \\
3_k &\equiv A_0 \cdot A_1 \cdot A_2 \cdots A_{i-2} \cdot c
\end{aligned} \tag{2.121c}$$

$$\begin{aligned}
&(\text{for } \ j \geqslant 2) \\
&i+1 \leqslant k \leqslant i+j-1: \qquad 1_k \equiv A_0 \cdot A_1 \cdot A_2 \cdots A_{i-2} \cdot a \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot A_i \cdot A_{i+1} \cdots A_{k-1} \\
&\qquad\qquad\qquad\qquad\qquad 2_k \equiv A_0 \cdot A_1 \cdot A_2 \cdots A_{i-2} \cdot b \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot A_i \cdot A_{i+1} \cdots A_{k-1} \\
&\qquad\qquad\qquad\qquad\qquad 3_k \equiv A_0 \cdot A_1 \cdot A_2 \cdots A_{i-2} \cdot c \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot A_i \cdot A_{i+1} \cdots A_{k-1}
\end{aligned} \tag{2.121d}$$

$$k = i + j: \quad 1_k \equiv A_0 \cdot A_1 \cdot A_2 \cdots A_{i-2} \underbrace{\cdot A_i \cdot A_{i+1} \cdots A_{i+j-2}}_{\text{for } j=1 \text{ it does NOT exist!}} \cdot X$$

$$2_k \equiv A_0 \cdot A_1 \cdot A_2 \cdots A_{i-2} \underbrace{\cdot A_i \cdot A_{i+1} \cdots A_{i+j-2}}_{\text{for } j=1 \text{ it does NOT exist!}} \cdot Y \tag{2.121e}$$

(for $j \geqslant 2$)

$$i + j + 1 \leqslant k \leqslant i + 2j - 1: \quad 1_k \equiv A_0 \cdot A_1 \cdot A_2 \cdots A_{i-2} \cdot A_i \cdot A_{i+1} \cdots A_{i+j-2} \cdot X \cdot A_{i+j} \cdot A_{i+j+1} \cdots A_{k-1}$$

$$2_k \equiv A_0 \cdot A_1 \cdot A_2 \cdots A_{i-2} \cdot A_i \cdot A_{i+1} \cdots A_{i+j-2} \cdot Y \cdot A_{i+j} \cdot A_{i+j+1} \cdots A_{k-1} \tag{2.121f}$$

$$k = i + 2j: \quad 1_k \equiv A_0 \cdot A_1 \cdot A_2 \cdots A_{i-2} \cdot \underbrace{A_i \cdot A_{i+1} \cdots A_{i+j-2}}_{\text{for } j=1 \text{ it does NOT exist!}} \cdot Z \cdot \underbrace{A_{i+j} \cdot A_{i+j+1} \cdots A_{i+2j-2}}_{\text{for } j=1 \text{ it does NOT exist!}} \tag{2.121g}$$

---

4 Now we write Table 2.13 even more compactly thus producing Table 2.14 exactly as Table 2.8 is obtained from Table 2.7. Table 2.14 has the same style as Table 2.8 but with some differences. In Table 2.14 we include only the part "*the units:*" of Table 2.13 since the other part of Table 2.13 remains the same for Table 2.14. The formulae of Table 2.14 are completely clarified if we compare them with the corresponding formulae of Table 2.13.

Table 2.14 A more compact form of Table 2.13. (The generalized version of Table 2.8 for random i, i.e., for $|f(i|j)|$).

$$Units\ for\ |f(i|j)|\ (nonrecursively):$$

$$the\ units:$$

$$k=0: \qquad (1_k, 2_k) = (1_0, 2_0) \tag{2.122a}$$

$$\begin{array}{l}(\text{for } \ i \geqslant 2) \\ 1 \leqslant k \leqslant i-1: \ \ (1_k, 2_k) \equiv \left[0, \prod_{m=0}^{k-1} A_m\right]\end{array} \tag{2.122b}$$

$$k=i: \qquad (1_k, 2_k, 3_k) \equiv \left(\prod_{m=0}^{i-2} A_m\right) \cdot (a,b,c) \tag{2.122c}$$

$$\begin{array}{l}(\text{for } \ j \geqslant 2) \\ i+1 \leqslant k \leqslant i+j-1: \qquad (1_k, 2_k, 3_k) \equiv \\ \qquad \left(\prod_{m=0}^{i-2} A_m\right) \cdot (a,b,c) \cdot \left(\prod_{m=i}^{k-1} A_m\right)\end{array} \tag{2.122d}$$

$$k=i+j: \quad (1_k, 2_k) \equiv \left(\prod_{m=0}^{i-2} A_m\right) \cdot \underbrace{\left(\prod_{m=i}^{i+j-2} A_m\right)}_{\text{for } j=1 \text{ it does NOT exist!}} \cdot (X,Y) \tag{2.122e}$$

(for $j \geqslant 2$)

$$i+j+1 \leqslant k \leqslant i+2j-1: \quad (1_k, 2_k) \equiv \left(\prod_{m=0}^{i-2} A_m\right) \cdot \left(\prod_{m=i}^{i+j-2} A_m\right)$$

$$\cdot (X,Y) \cdot \left(\prod_{m=i+j}^{k-1} A_m\right) \tag{2.122f}$$

$$k = i+2j: \qquad 1_k \equiv \left(\prod_{m=0}^{i-2} A_m\right) \cdot \underbrace{\left(\prod_{m=i}^{i+j-2} A_m\right)}_{\text{for } j=1 \text{ it does NOT exist!}} \cdot$$

$$Z \qquad \cdot \underbrace{\left(\prod_{m=i+j}^{i+2j-2} A_m\right)}_{\text{for } j=1 \text{ it does NOT exist!}} \tag{2.122g}$$

2.9 Determining how the lines $|f(i|j)|$ are composed of the basic units $1_k, 2_k, 3_k$.

1 Now, wishing to see how the basic units $1_k, 2_k, 3_k$ form the lines $|f(i|j)|$, we are doing for $|f(i|j)|$ what we have done in Sec. 2.6 for $|f(1|j)|$. We start by presenting the examples $|f(2|j)|$ and $|f(3|j)|$ [the example $|f(1|j)|$ has already been presented in Sec. 2.6, see (2.83)–(2.91)]. For doing the examples we may have in mind, for help, (2.38)–(2.40) as well as (2.100)–(2.105). The notation and the manner in which we work are familiar from Sec. 2.6.

We write directly the corresponding to (2.86)–(2.91) formulae [which were for $|f(1|j)|$] for $|f(2|j)|$ and $|f(3|j)|$.

First we write the formulae for the example $|f(2|j)|$:

$$(i=2, j=1, \quad \textit{period} \;\; 2^4)$$

$$\begin{array}{ll}|f(2|1)|_{k=0} = & 1(2\cdot 1_0 2\cdot 2_0)1(2\cdot 2_0 2\cdot 2_0)1(2\cdot 2_0 2\cdot 2_0)1(2\cdot 2_0 2\cdot 1_0)\\ |f(2|1)|_{k=1} = & 1(1_1 2_1)1(2_1 2_1)1(2_1 2_1)1(2_1 1_1)\\ |f(2|1)|_{k=2} = & 1(1_2 2_2)1(2_2 3_2)\\ |f(2|1)|_{k=3} = & 1(1_3)1(2_3)\\ |f(2|1)|_{k=4} = & 1(1_4)\end{array} \tag{2.123}$$

$$(i=2, j=2, \quad period\ \ 2^6)$$

$$\begin{array}{ll}|f(2|2)|_{k=0} = & 2[\,2(2\cdot 1_0 2\cdot 2_0)2(2\cdot 2_0 2\cdot 2_0)\,]\\ & 2[\,2(2\cdot 2_0 2\cdot 2_0)2(2\cdot 2_0 2\cdot 1_0)\,]\\ |f(2|2)|_{k=1} = & 2[\,2(1_1 2_1)2(2_1 2_1)\,]2[\,2(2_1 2_1)2(2_1 1_1)\,]\\ |f(2|2)|_{k=2} = & 2[\,2(1_2)2(2_2)\,]2[\,2(2_2)2(3_2)\,]\\ |f(2|2)|_{k=3} = & 2[\,1(1_3)1(2_3)\,]2[\,1(2_3)1(3_3)\,]\\ |f(2|2)|_{k=4} = & 2(1_4)2(2_4)\\ |f(2|2)|_{k=5} = & 1(1_5)1(2_5)\\ |f(2|2)|_{k=6} = & 1(1_6)\end{array} \tag{2.124}$$

$$(i=2, j=3, \quad period\ \ 2^8)$$

$$\begin{array}{ll}|f(2|3)|_{k=0} = & 4[\,4(2\cdot 1_0 2\cdot 2_0)4(2\cdot 2_0 2\cdot 2_0)\,]\\ & 4[\,4(2\cdot 2_0 2\cdot 2_0)4(2\cdot 2_0 2\cdot 1_0)\,]\\ |f(2|3)|_{k=1} = & 4[\,4(1_1 2_1)4(2_1 2_1)\,]4[\,4(2_1 2_1)4(2_1 1_1)\,]\\ |f(2|3)|_{k=2} = & 4[\,4(1_2)4(2_2)\,]4[\,4(2_2)4(3_2)\,]\\ |f(2|3)|_{k=3} = & 4[\,2(1_3)2(2_3)\,]4[\,2(2_3)2(3_3)\,]\\ |f(2|3)|_{k=4} = & 4[\,1(1_4)1(2_4)\,]4[\,1(2_4)1(3_4)\,]\\ |f(2|3)|_{k=5} = & 4(1_5)4(2_5)\\ |f(2|3)|_{k=6} = & 2(1_6)2(2_6)\\ |f(2|3)|_{k=7} = & 1(1_7)1(2_7)\\ |f(2|3)|_{k=8} = & 1(1_8)\end{array} \tag{2.125}$$

In the above formulae when we write, e.g., $(2\cdot 1_0 2\cdot 2_0)$ we evidently mean $1_0 1_0 2_0 2_0$.

Next we write again (2.123)–(2.125) using the convention with the asterisks in accordance with (2.89)–(2.91):

$$(i=2, j=1, \quad period\ \ 2^4)$$

$$
\begin{aligned}
|f(2|1)|_{k=0} = \;& \{[\,(1_0*)(2_0*)\,][\,(2_0*)(2_0*)\,]\}\{[\,(2_0*)(2_0*)\,][\,(2_0*)(1_0*)\,]\}\\
|f(2|1)|_{k=1} = \;& (1_1 2_1)(2_1 2_1)(2_1 2_1)(2_1 1_1)\\
|f(2|1)|_{k=2} = \;& (1_2 2_2)(2_2 3_2)\\
|f(2|1)|_{k=3} = \;& (1_3)(2_3)\\
|f(2|1)|_{k=4} = \;& (1_4)
\end{aligned}
\tag{2.126}
$$

$$(i = 2, j = 2, \quad period \;\; 2^6)$$

$$
\begin{aligned}
|f(2|2)|_{k=0} = \;& \{[\,(1_0*)(2_0*)*\,][\,(2_0*)(2_0*)*\,]*\}\\
& \{[\,(2_0*)(2_0*)*\,][\,(2_0*)(1_0*)*\,]*\}\\
|f(2|2)|_{k=1} = \;& [\,(1_1 2_1*)(2_1 2_1*)*\,][\,(2_1 2_1*)(2_1 1_1*)*\,]\\
|f(2|2)|_{k=2} = \;& [\,(1_2*)(2_2*)*\,][\,(2_2*)(3_2*)*\,]\\
|f(2|2)|_{k=3} = \;& [\,(1_3)(2_3)*\,][\,(2_3)(3_3)*\,]\\
|f(2|2)|_{k=4} = \;& (1_4*)(2_4*)\\
|f(2|2)|_{k=5} = \;& (1_5)(2_5)\\
|f(2|2)|_{k=6} = \;& (1_6)
\end{aligned}
\tag{2.127}
$$

$$(i = 2, j = 3, \quad period \;\; 2^8)$$

$$
\begin{aligned}
|f(2|3)|_{k=0} = \;& \{[\,(1_0*)(2_0*)**\,][\,(2_0*)(2_0*)**\,]**\}\\
& \{[\,(2_0*)(2_0*)**\,][\,(2_0*)(1_0*)**\,]**\}\\
|f(2|3)|_{k=1} = \;& [\,(1_1 2_1 **)(2_1 2_1 **)**\,][\,(2_1 2_1 **)(2_1 1_1 **)**\,]\\
|f(2|3)|_{k=2} = \;& [\,(1_2 **)(2_2 **)**\,][\,(2_2 **)(3_2 **)**\,]\\
|f(2|3)|_{k=3} = \;& [\,(1_3*)(2_3*)**\,][\,(2_3*)(3_3*)**\,]\\
|f(2|3)|_{k=4} = \;& [\,(1_4)(2_4)**\,][\,(2_4)(3_4)**\,]\\
|f(2|3)|_{k=5} = \;& (1_5 **)(2_5 **)\\
|f(2|3)|_{k=6} = \;& (1_6*)(2_6*)\\
|f(2|3)|_{k=7} = \;& (1_7)(2_7)\\
|f(2|3)|_{k=8} = \;& (1_8)
\end{aligned}
\tag{2.128}
$$

Now we write the formulae for the example $|f(3|j)|$:

$$(i = 3, j = 1, \quad period \;\; 2^5)$$

$$\begin{array}{ll}
|f(3|1)|_{k=0} = & 1(4\cdot 1_0 4\cdot 2_0)1(4\cdot 2_0 4\cdot 2_0)1(4\cdot 2_0 4\cdot 2_0)1(4\cdot 2_0 4\cdot 1_0) \\
|f(3|1)|_{k=1} = & 1(2\cdot 1_1 2\cdot 2_1)1(2\cdot 2_1 2\cdot 2_1)1(2\cdot 2_1 2\cdot 2_1)1(2\cdot 2_1 2\cdot 1_1) \\
|f(3|1)|_{k=2} = & 1(1_2 2_2)1(2_2 2_2)1(2_2 2_2)1(2_2 1_2) \\
|f(3|1)|_{k=3} = & 1(1_3 2_3)1(2_3 3_3) \\
|f(3|1)|_{k=4} = & 1(1_4)1(2_4) \\
|f(3|1)|_{k=5} = & 1(1_5)
\end{array} \tag{2.129}$$

$$(i=3, j=2, \quad period\ \ 2^7)$$

$$\begin{array}{ll}
|f(3|2)|_{k=0} = & 2[\,2(4\cdot 1_0 4\cdot 2_0)2(4\cdot 2_0 4\cdot 2_0)\,] \\
 & 2[\,2(4\cdot 2_0 4\cdot 2_0)2(4\cdot 2_0 4\cdot 1_0)\,] \\
|f(3|2)|_{k=1} = & 2[\,2(2\cdot 1_1 2\cdot 2_1)2(2\cdot 2_1 2\cdot 2_1)\,] \\
 & 2[\,2(2\cdot 2_1 2\cdot 2_1)2(2\cdot 2_1 2\cdot 1_1)\,] \\
|f(3|2)|_{k=2} = & 2[\,2(1_2 2_2)2(2_2 2_2)\,]2[\,2(2_2 2_2)2(2_2 1_2)\,] \\
|f(3|2)|_{k=3} = & 2[\,2(1_3)2(2_3)\,]2[\,2(2_3)2(3_3)\,] \\
|f(3|2)|_{k=4} = & 2[\,1(1_4)1(2_4)\,]2[\,1(2_4)1(3_4)\,] \\
|f(3|2)|_{k=5} = & 2(1_5)2(2_5) \\
|f(3|2)|_{k=6} = & 1(1_6)1(2_6) \\
|f(3|2)|_{k=7} = & 1(1_7)
\end{array} \tag{2.130}$$

$$(i=3, j=3, \quad period\ \ 2^9)$$

$$\begin{array}{ll}
|f(3|3)|_{k=0} = & 4[\,4(4\cdot 1_0 4\cdot 2_0)4(4\cdot 2_0 4\cdot 2_0)\,] \\
 & 4[\,4(4\cdot 2_0 4\cdot 2_0)4(4\cdot 2_0 4\cdot 1_0)\,] \\
|f(3|3)|_{k=1} = & 4[\,4(2\cdot 1_1 2\cdot 2_1)4(2\cdot 2_1 2\cdot 2_1)\,] \\
 & 4[\,4(2\cdot 2_1 2\cdot 2_1)4(2\cdot 2_1 2\cdot 1_1)\,] \\
|f(3|3)|_{k=2} = & 4[\,4(1_2 2_2)4(2_2 2_2)\,]4[\,4(2_2 2_2)4(2_2 1_2)\,] \\
|f(3|3)|_{k=3} = & 4[\,4(1_3)4(2_3)\,]4[\,4(2_3)4(3_3)\,] \\
|f(3|3)|_{k=4} = & 4[\,2(1_4)2(2_4)\,]4[\,2(2_4)2(3_4)\,] \\
|f(3|3)|_{k=5} = & 4[\,1(1_5)1(2_5)\,]4[\,1(2_5)1(3_5)\,] \\
|f(3|3)|_{k=6} = & 4(1_6)4(2_6) \\
|f(3|3)|_{k=7} = & 2(1_7)2(2_7) \\
|f(3|3)|_{k=8} = & 1(1_8)1(2_8) \\
|f(3|3)|_{k=9} = & 1(1_9)
\end{array} \tag{2.131}$$

Next we write (2.129)–(2.131) using the convention with the asterisks, as we did for the previous example $|f(2|j)|$:

$$(i=3, j=1, \quad period \;\; 2^5)$$

$$\begin{array}{ll} |f(3|1)|_{k=0} = & \{[\,(1_0**)(2_0**)\,][\,(2_0**)(2_0**)\,]\} \\ & \{[\,(2_0**)(2_0**)\,][\,(2_0**)(1_0**)\,]\} \\ |f(3|1)|_{k=1} = & \{[\,(1_1*)(2_1*)\,][\,(2_1*)(2_1*)\,]\} \\ & \{[\,(2_1*)(2_1*)\,][\,(2_1*)(1_1*)\,]\} \\ |f(3|1)|_{k=2} = & (1_2 2_2)(2_2 2_2)(2_2 2_2)(2_2 1_2) \\ |f(3|1)|_{k=3} = & (1_3 2_3)(2_3 3_3) \\ |f(3|1)|_{k=4} = & (1_4)(2_4) \\ |f(3|1)|_{k=5} = & (1_5) \end{array} \tag{2.132}$$

$$(i=3, j=2, \quad period \;\; 2^7)$$

$$\begin{array}{ll} |f(3|2)|_{k=0} = & \{[\,(1_0**)(2_0**)*][\,(2_0**)(2_0**)*]*\} \\ & \{[\,(2_0**)(2_0**)*][\,(2_0**)(1_0**)*]*\} \\ |f(3|2)|_{k=1} = & \{[\,(1_1*)(2_1*)*][\,(2_1*)(2_1*)*]*\} \\ & \{[\,(2_1*)(2_1*)*][\,(2_1*)(1_1*)*]*\} \\ |f(3|2)|_{k=2} = & [\,(1_2 2_2*)(2_2 2_2*)*][\,(2_2 2_2*)(2_2 1_2*)*] \\ |f(3|2)|_{k=3} = & [\,(1_3*)(2_3*)*][\,(2_3*)(3_3*)*] \\ |f(3|2)|_{k=4} = & [\,(1_4)(2_4)*][\,(2_4)(3_4)*] \\ |f(3|2)|_{k=5} = & (1_5*)(2_5*) \\ |f(3|2)|_{k=6} = & (1_6)(2_6) \\ |f(3|2)|_{k=7} = & (1_7) \end{array} \tag{2.133}$$

$$(i=3, j=3, \quad period \;\; 2^9)$$

$$\begin{aligned}
|f(3|3)|_{k=0} = \;& \{[\,(1_0 **)(2_0 **)**\,][\,(2_0 **)(2_0 **)**\,]**\} \\
& \{[\,(2_0 **)(2_0 **)**\,][\,(2_0 **)(1_0 **)**\,]**\} \\
|f(3|3)|_{k=1} = \;& \{[\,(1_1 *)(2_1 *)**\,][\,(2_1 *)(2_1 *)**\,]**\} \\
& \{[\,(2_1 *)(2_1 *)**\,][\,(2_1 *)(1_1 *)**\,]**\} \\
|f(3|3)|_{k=2} = \;& [\,(1_2 2_2 **)(2_2 2_2 **)**\,][\,(2_2 2_2 **)(2_2 1_2 **)**\,] \\
|f(3|3)|_{k=3} = \;& [\,(1_3 **)(2_3 **)**\,][\,(2_3 **)(3_3 **)**\,] \\
|f(3|3)|_{k=4} = \;& [\,(1_4 *)(2_4 *)**\,][\,(2_4 *)(3_4 *)**\,] \\
|f(3|3)|_{k=5} = \;& [\,(1_5)(2_5)**\,][\,(2_5)(3_5)**\,] \\
|f(3|3)|_{k=6} = \;& (1_6 **)(2_6 **) \\
|f(3|3)|_{k=7} = \;& (1_7 *)(2_7 *) \\
|f(3|3)|_{k=8} = \;& (1_8)(2_8) \\
|f(3|3)|_{k=9} = \;& (1_9)
\end{aligned} \tag{2.134}$$

2 As we have produced Table 2.9 for $|f(1|j)|$, we produce now Table 2.15 for $|f(i|j)|$. The procedure by which Table 2.15 is obtained is same with the procedure for obtaining Table 2.9, and the notation is exactly same as well. So we present directly Table 2.15 without further descriptions:

Table 2.15 How the basic units $1_k, 2_k, 3_k$ form the line $|f(i|j)|$ for random i and j and for various values of k. (The generalized version of Table 2.9 for random i, i.e., for $|f(i|j)|$).

General formulae for line $|f(i|j)|$:

$$(k \in \{0, 1, 2, \cdots, 2j+i\}, \quad \textit{period} \;\; 2^{2j+i}) \tag{2.135}$$

$$\begin{aligned}
|f(i|j)|_{0\leqslant k\leqslant i-2} = \;& 2^{j-1}[\,2^{j-1}(2^{i-1-k}\cdot 1_k \; 2^{i-1-k}\cdot 2_k) \\
& \qquad 2^{j-1}(2^{i-1-k}\cdot 2_k \; 2^{i-1-k}\cdot 2_k)\,] \\
& 2^{j-1}[\,2^{j-1}(2^{i-1-k}\cdot 2_k \; 2^{i-1-k}\cdot 2_k) \\
& \qquad 2^{j-1}(2^{i-1-k}\cdot 2_k \; 2^{i-1-k}\cdot 1_k)\,]
\end{aligned}$$

$$
\begin{aligned}
|f(i|j)|_{k=i-1} = \quad & 2^{j-1}[\, 2^{j-1}(2^{i-1-k}\cdot 1_k\ 2^{i-1-k}\cdot 2_k) \\
& \quad 2^{j-1}(2^{i-1-k}\cdot 2_k\ 2^{i-1-k}\cdot 2_k)\,] \\
& 2^{j-1}[\, 2^{j-1}(2^{i-1-k}\cdot 2_k\ 2^{i-1-k}\cdot 2_k) \\
& \quad 2^{j-1}(2^{i-1-k}\cdot 2_k\ 2^{i-1-k}\cdot 1_k)\,] \\
|f(i|j)|_{(i-1)+1\leqslant k\leqslant (i-1)+j} = \quad & 2^{j-1}[\, 2^{(i-1)+j-k}(1_k)\ 2^{(i-1)+j-k}(2_k)\,] \\
& 2^{j-1}[\, 2^{(i-1)+j-k}(2_k)\ 2^{(i-1)+j-k}(3_k)\,] \\
|f(i|j)|_{(i-1)+j+1\leqslant k\leqslant (i-1)+2j} = \quad & [\, 2^{(i-1)+2j-k}(1_k)\ 2^{(i-1)+2j-k}(2_k)\,] \\
|f(i|j)|_{k=(i-1)+2j+1=i+2j} = \quad & (1_k)
\end{aligned}
\tag{2.136}
$$

or equivalently

$$
\begin{aligned}
|f(i|j)|_{0\leqslant k\leqslant i-2} = \quad & \{[\,(1_k(i-1-k)*)(2_k(i-1-k)*)(j-1)*\,] \\
& \ [\,(2_k(i-1-k)*)(2_k(i-1-k)*)(j-1)*\,](j-1)*\} \\
& \{[\,(2_k(i-1-k)*)(2_k(i-1-k)*)(j-1)*\,] \\
& \ [\,(2_k(i-1-k)*)(1_k(i-1-k)*)(j-1)*\,](j-1)*\}
\end{aligned}
$$

$$
\begin{aligned}
|f(i|j)|_{k=i-1} = \quad & \{[\,(1_k(i-1-k)*)(2_k(i-1-k)*)(j-1)*\,] \\
& \ [\,(2_k(i-1-k)*)(2_k(i-1-k)*)(j-1)*\,](j-1)*\} \\
& \{[\,(2_k(i-1-k)*)(2_k(i-1-k)*)(j-1)*\,] \\
& \ [\,(2_k(i-1-k)*)(1_k(i-1-k)*)(j-1)*\,](j-1)*\}
\end{aligned}
$$

$$
\begin{aligned}
|f(i|j)|_{(i-1)+1\leqslant k\leqslant (i-1)+j} = \quad & \{[\,(1_k[\,(i-1)+j-k\,]*) \\
& \quad (2_k[\,(i-1)+j-k\,]*)(j-1)*\,] \\
& \ [\,(2_k[\,(i-1)+j-k\,]*) \\
& \quad (3_k[\,(i-1)+j-k\,]*)(j-1)*\,]\}
\end{aligned}
$$

$$
\begin{aligned}
|f(i|j)|_{(i-1)+j+1\leqslant k\leqslant (i-1)+2j} = \quad & [\,(1_k[\,(i-1)+2j-k\,]*) \\
& \ (2_k[\,(i-1)+2j-k\,]*)\,]
\end{aligned}
$$

$$|f(i|j)|_{k=(i-1)+2j+1=i+2j} = \quad (1_k)$$

(2.137)

We can write Table 2.15 more compactly as Table 2.16.

Table 2.16 A more compact version of Table 2.15.

$$General\ formulae\ for\ line\ |f(i|j)|:$$

$$(k \in \{0,1,2,\cdots,i+2j\}, \quad period\ \ 2^{i+2j}) \tag{2.138}$$

$$\begin{aligned}|f(i|j)|_{0\leqslant k\leqslant i-1} = \quad & 2^{j-1}[\,2^{j-1}(2^{i-1-k}\cdot 1_k\ 2^{i-1-k}\cdot 2_k) \\ & \quad\ 2^{j-1}(2^{i-1-k}\cdot 2_k\ 2^{i-1-k}\cdot 2_k)\,] \\ & 2^{j-1}[\,2^{j-1}(2^{i-1-k}\cdot 2_k\ 2^{i-1-k}\cdot 2_k) \\ & \quad\ 2^{j-1}(2^{i-1-k}\cdot 2_k\ 2^{i-1-k}\cdot 1_k)\,]\end{aligned}$$

$$\begin{aligned}|f(i|j)|_{i\leqslant k\leqslant i+j-1} = \quad & 2^{j-1}[\,2^{i+j-1-k}(1_k)\ 2^{i+j-1-k}(2_k)\,] \\ & 2^{j-1}[\,2^{i+j-1-k}(2_k)\ 2^{i+j-1-k}(3_k)\,]\end{aligned}$$

$$|f(i|j)|_{i+j\leqslant k\leqslant i+2j-1} = [\,2^{i+2j-1-k}(1_k)\ 2^{i+2j-1-k}(2_k)\,]$$

$$|f(i|j)|_{k=i+2j} = (1_k)$$

(2.139)

or equivalently

$$
\begin{aligned}
|f(i|j)|_{0\leqslant k\leqslant i-1} = \quad & \{[\,(1_k(i-1-k)*)(2_k(i-1-k)*)(j-1)*\,] \\
& \;[\,(2_k(i-1-k)*)(2_k(i-1-k)*)(j-1)*\,](j-1)*\} \\
& \{[\,(2_k(i-1-k)*)(2_k(i-1-k)*)(j-1)*\,] \\
& \;[\,(2_k(i-1-k)*)(1_k(i-1-k)*)(j-1)*\,](j-1)*\}
\end{aligned}
$$

$$
\begin{aligned}
|f(i|j)|_{i\leqslant k\leqslant i+j-1} = \quad & \{[\,(1_k(i+j-1-k)*) \\
& \;\;(2_k(i+j-1-k)*)(j-1)*\,] \\
& \;[\,(2_k(i+j-1-k)*) \\
& \;\;(3_k(i+j-1-k)*)(j-1)*\,]\}
\end{aligned}
$$

$$
\begin{aligned}
|f(i|j)|_{i+j\leqslant k\leqslant i+2j-1} = \quad & [\,(1_k(i+2j-1-k)*) \\
& \;(2_k(i+2j-1-k)*)\,]
\end{aligned}
$$

$$
|f(i|j)|_{k=i+2j} = (1_k)
$$

$$(2.140)$$

---

3 As we did the example $|f(1|4)|$ in (2.99) for checking the general formulae of Table 2.9 we do now the example of $|f(4|4)|$ for checking the general formulae of Table 2.16. We work in a similar manner, with the same notation, as in Sec. 2.6,Subsec. 5 for $|f(1|4)|$.

Analytically, knowing the general form of line $|f(i|j)|$ [see Chapters 5,6,7 of the *Mathematical Diary 1993–1998* but also the examples (2.38)–(2.40) and so on] we can easily see that

$$
|f(4|4)| = 8[\,8(8{\cdot}0\;8{\cdot}1)8(8{\cdot}1\;8{\cdot}1)\,]8[\,8(8{\cdot}1\;8{\cdot}1)8(8{\cdot}1\;8{\cdot}0)\,] \quad (2.141)
$$

with period 2^{12}. Thus we obtain easily [working as in Sec. 2.6,5—see (2.98) and (2.99)] that

$$
(i=4, j=4, \quad \textit{period}\;\; 2^{12})
$$

$$
\begin{aligned}
|f(4|4)|_{k=0} = \;& 8[\,8(8\cdot 1_0\; 8\cdot 2_0)8(8\cdot 2_0\; 8\cdot 2_0)\,]\\
& 8[\,8(8\cdot 2_0\; 8\cdot 2_0)8(8\cdot 2_0\; 8\cdot 1_0)\,]\\
|f(4|4)|_{k=1} = \;& 8[\,8(4\cdot 1_1\; 4\cdot 2_1)8(4\cdot 2_1\; 4\cdot 2_1)\,]\\
& 8[\,8(4\cdot 2_1\; 4\cdot 2_1)8(4\cdot 2_1\; 4\cdot 1_1)\,]\\
|f(4|4)|_{k=2} = \;& 8[\,8(2\cdot 1_2\; 2\cdot 2_2)8(2\cdot 2_2\; 2\cdot 2_2)\,]\\
& 8[\,8(2\cdot 2_2\; 2\cdot 2_2)8(2\cdot 2_2\; 2\cdot 1_2)\,]\\
|f(4|4)|_{k=3} = \;& 8[\,8(1\cdot 1_3\; 1\cdot 2_3)8(1\cdot 2_3\; 1\cdot 2_3)\,]\\
& 8[\,8(1\cdot 2_3\; 1\cdot 2_3)8(1\cdot 2_3\; 1\cdot 1_3)\,]\\
\\
|f(4|4)|_{k=4} = \;& 8[\,8\cdot 1_4\; 8\cdot 2_4\,]8[\,8\cdot 2_4\; 8\cdot 3_4\,]\\
|f(4|4)|_{k=5} = \;& 8[\,4\cdot 1_5\; 4\cdot 2_5\,]8[\,4\cdot 2_5\; 4\cdot 3_5\,]\\
|f(4|4)|_{k=6} = \;& 8[\,2\cdot 1_6\; 2\cdot 2_6\,]8[\,2\cdot 2_6\; 2\cdot 3_6\,]\\
|f(4|4)|_{k=7} = \;& 8[\,1\cdot 1_7\; 1\cdot 2_7\,]8[\,1\cdot 2_7\; 1\cdot 3_7\,]\\
\\
|f(4|4)|_{k=8} = \;& [\,8\cdot 1_8\; 8\cdot 2_8\,]\\
|f(4|4)|_{k=9} = \;& [\,4\cdot 1_9\; 4\cdot 2_9\,]\\
|f(4|4)|_{k=10} = \;& [\,2\cdot 1_{10}\; 2\cdot 2_{10}\,]\\
|f(4|4)|_{k=11} = \;& [\,1\cdot 1_{11}\; 1\cdot 2_{11}\,]\\
\\
|f(4|4)|_{k=12} = \;& (1_{12})
\end{aligned}
\tag{2.142}
$$

Now using the example (2.142) for checking the general formulae of Table 2.16 we can, indeed, see that (2.142) is produced directly from (2.139)) if we put $i = j = 4$.

Here the basic goal of the present chapter (i.e., Chapter 2), which was to determine the analytical form of the general line $2P^{\,i}_{j}$ or equivalently $|f(i|j)|$, has been accomplished. Essentially the basic work of the chapter begins in Sec. 2.3.

The basic results of the chapter, concerning the form of the general line $|f(i|j)|$, appear in:

—*Table 2.10 [recursive algorithm for determining the basic units of lines $|f(i|j)|$];*

—*Tables 2.13 and 2.14 [nonrecursive algorithm for determining the basic units of lines $|f(i|j)|$];*

—*Table 2.16 [how the basic units form the lines $|f(i|j)|$].*

Having at our disposal the analytical form of the lines $2P\,^{i}_{j}$, which are the lines $|f(i|j)|$, it is easier to determine the superpositions of these lines. This will be done in the following chapters.

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Chapter 3

Superposing the lines $|f(i|j)|$. Technical details.

Our main goal is to use the previously developed machinery for determining the superposition of the lines $|f(i|j)|$. But before this, some further work must be done concerning the technical details of how exactly this superposition is performed. Then we can proceed and produce $|f(1|1,2,\cdots,j)|$ and so on.

3.1 The examples of $|f(1|1)|$ and $|f(1|1,2)|$.

1 In the following when we represent line $|f(i|j)|$ we prefer to work with the *maximal* k ($k = i + 2j$) for each case because then, as we see in Tables 2.15 and 2.16, the line is represented by *just one unit*, that is, unit 1_k which is taken from Tables 2.10, 2.11, 2.13, and 2.14. So the manipulations get simpler.

2 Let us see the example of $|f(1|1)|$.

From Tables 2.10, 2.11, 2.13–2.16 (or from their partial forms for $i = 1$, that is, Tables 2.3 and 2.5–2.9) we obtain directly the following results.

Since it is $i = j = 1$ for the maximal k we have $k = i + 2j = 3$. So it is $1_k = 1_3 \equiv Z$ and since, here, it is

$$Z \equiv [\, 2^{2^2+2^1} + 2^{2^2+2^0} + 2^{2^1+2^0} + 2^{2^2} + 2^{2^1} + 2^{2^0} \,] \tag{3.1}$$

we obtain

$$\begin{aligned} |f(1|1)| = 1_3 &= [\, 2^{2^2+2^1} + 2^{2^2+2^0} + 2^{2^1+2^0} + 2^{2^2} + 2^{2^1} + 2^{2^0} \,] \\ &= (2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 2^1), \end{aligned} \tag{3.2}$$

with period 2^{i+2j} which is 2^3. Here and in the following always we accept the convention (2.29) or (2.67) for the strings l, that is,

$$l \in \{0, 1, 2, \cdots, 2^{2^k} - 1\}. \tag{3.3}$$

3 If we wish to find $|f(1|1,2)|$ we must take $|f(1|1)|$ four times and then superpose it with $|f(1|2)|$ (see Chapter 7 of the *Mathematical Diary 1993–1998* as well as Sec. 2.1,Subsec. 4). Thus we must know $|f(1|2)|$. From the tables of the previous chapter we obtain the following results.

Since $i = 1$ and $j = 2$ we have for the maximal k that $k = i+2j = 5$. Also it is $1_k = 1_5 \equiv A_1 \cdot Z \cdot A_3$ with

$$A_1 \equiv (2^{2^1} + 1), \quad A_3 \equiv (2^{2^3} + 1),$$
$$Z \equiv [\, 2^{2^4+2^2} + 2^{2^4+2^0} + 2^{2^2+2^0} + 2^{2^4} + 2^{2^2} + 2^{2^0} \,]. \tag{3.4}$$

Thus it is

$$|f(1|2)| = 1_5 =$$
$$(2^{2^1} + 1) \cdot [\, 2^{2^4+2^2} + 2^{2^4+2^0} + 2^{2^2+2^0} + 2^{2^4} + 2^{2^2} + 2^{2^0} \,] \cdot (2^{2^3} + 1), \tag{3.5}$$

with period 2^{i+2j} which is 2^5.

But for finding $|f(1|1, 2)|$, as we have said, we must take $|f(1|1)|$ four times, that is, we must take $4|f(1|1)|$. How this is analytically performed will be described in the next sections.

Note 3.1 The lines $|f(i|j)|$ are expressed as functions of exponentials of two. This leads us to a binary representation of the lines (we have already noticed some relevant things in Sec. 2.2) in the sense that we may write the exponentials in increasing order of the exponents and have factors 1 or 0 in front of existing or nonexisting exponentials respectively precisely as we do when we represent integers in the binary system. In the following sections we are going to see all these analytically. □

3.2 How from line $|f(i|j)|$ we obtain its multiple $2|f(i|j)|$ which is line $|f(i|j)|$ taken twice.

1 Wishing to perform superpositions of lines $|f(i|j)|$ it is necessary to see how from line $|f(i|j)|$, whose form is known from the tables of Chapter 2, we obtain its multiples $2|f(i|j)|, 4|f(i|j)|, \cdots, 2^n|f(i|j)|$ which are line $|f(i|j)|$ taken for two, four,$\cdots$, and generally for 2^n times respectively. All these will be considered here in detail.

Let us see first how from line $|f(i|j)|$ we produce its multiple $2|f(i|j)|$ which is line $|f(i|j)|$ taken twice. The results will be easily generalized for the other multiples as well.

We work, for simplicity, with the maximal k and, as we see from Tables 2.15 or 2.16, for this maximal k it is $k = i+2j$ and $|f(i|j)| = 1_k$. We know that

$$1_k \in \{0, 1, 2, \cdots, 2^{2^k} - 1\}, \tag{3.6}$$

hence for $k = i + 2j$ it is

$$\begin{aligned} |f(i|j)| = 1_k &\in \{0, 1, 2, \cdots, 2^{2^k} - 1\} \\ &= \{0, 1, 2, \cdots, 2^{2^{i+2j}} - 1\}. \end{aligned} \tag{3.7}$$

We consider now $2|f(i|j)|$ which is $2^1|f(i|j)|$. From Sections 1.1 and 1.2 we know that if for $|f(i|j)|$ we work for k then for its doubled counterpart $2|f(i|j)|$, which is the concatenation of $|f(i|j)|$ with itself i.e. $|f(i|j)||f(i|j)|$, we work for k' with $k' = k + 1$. So it is

$$k' = k + 1 = i + 2j + 1, \tag{3.8}$$

and

$$\begin{aligned} 2|f(i|j)| &\in \{0, 1, 2, \cdots, 2^{2^{k'}} - 1\} \\ &= \{0, 1, 2, \cdots, 2^{2^{k+1}} - 1\} \\ &= \{0, 1, 2, \cdots, 2^{2^{i+2j+1}} - 1\}. \end{aligned} \tag{3.9}$$

Knowing $|f(i|j)|$ we obtain $2|f(i|j)|$ by the following procedure. Since it is

$$|f(i|j)| \in \{0, 1, 2, \cdots, 2^{2^{i+2j}} - 1\}, \tag{3.10}$$

we may write the set in (3.10) as

$$\{0, 1, 2, \cdots, |f(i|j)|, \cdots, 2^{2^{i+2j}} - 1\}, \tag{3.11}$$

and of course in (3.11) $|f(i|j)|$ may equally take the first values 0,1,2 or the last value $2^{2^{i+2j}} - 1$. Because of (3.10) and (3.11) the concatenation of $|f(i|j)|$ with itself i.e. $|f(i|j)||f(i|j)|$ i.e. $2|f(i|j)|$ belongs to the set

$$\begin{array}{llllllll} \{ & \underline{00} & 01 & 02 & \cdots & 0|f| & \cdots & 0(2^{2^{i+2j}} - 1) \\ & 10 & \underline{11} & 12 & \cdots & 1|f| & \cdots & 1(2^{2^{i+2j}} - 1) \\ & 20 & 21 & \underline{22} & \cdots & 2|f| & \cdots & 2(2^{2^{i+2j}} - 1) \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & |f|0 & |f|1 & |f|2 & \cdots & \underline{\underline{|f||f|}} & \cdots & |f|(2^{2^{i+2j}} - 1) \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

$$(2^{2^{i+2j}}-1)0,\ \ (2^{2^{i+2j}}-1)1,\ \ (2^{2^{i+2j}}-1)2,\ \ \cdots,\ \ (2^{2^{i+2j}}-1)|f|,\ \ \cdots,$$
$$\underline{(2^{2^{i+2j}}-1)(2^{2^{i+2j}}-1)}\ \ \} \tag{3.12}$$

The set in (3.12) is written under the form of an array. It is read row-by-row and we omit the commas between the members except for the last row. Each member is concatenation of two strings, for example, 12 is string 1 concatenated with string 2, $1|f|$ is string 1 concatenated with string $|f|$, $(2^{2^{i+2j}}-1)2$ is string $(2^{2^{i+2j}}-1)$ concatenated with string 2, and so on. Clearly $2|f(i|j)|$ occupies the diagonal elements 00,11,22,$\cdots$ in the array and because of this we underline these elements (especially $|f||f|$ is underlined twice).

2 Set (3.12) may be written in just one row taking the form

$$\{00, 01, 02, \cdots, |f||f|, \cdots, (2^{2^{i+2j}}-1)(2^{2^{i+2j}}-1)\}. \tag{3.13}$$

Using the practice, described in Sec. 1.1 and Sec. 1.2 [compare also array (1.23) with array (3.12) having in mind that for the strings we accept now convention (2.29) instead of (2.28) and that $k = i+2j$], of renaming the strings by their natural enumeration we present the elements of set (3.13) by this enumeration and, since the set contains totally $2^{2^{i+2j+1}}$ elements ($2^{2^{i+2j}} \cdot 2^{2^{i+2j}} = 2^{2^{i+2j+1}}$), the new form of the set is

$$\{0, 1, 2, \cdots, 2|f(i|j)|, \cdots, 2^{2^{i+2j+1}}-1\}. \tag{3.14}$$

Thus we may write

$$2|f(i|j)| \in \{0, 1, 2, \cdots, 2^{2^{i+2j+1}}-1\}$$
$$= \{0, 1, 2, \cdots, 2^{2^{k'}}-1\}. \tag{3.15}$$

Since we have represented $2|f(i|j)|$, which is $|f(i|j)||f(i|j)|$, as a diagonal element of array (3.12) let us see analytically these diagonal elements and their position in the set of (3.14) or (3.15):

00 is 0, that is $0 \cdot 2^{2^{i+2j}} + 1 - 1$, in set $\{0, 1, 2, \cdots, 2^{2^{k'}}-1\}$,

11 is $1 \cdot 2^{2^{i+2j}} + 2 - 1$ in set $\{0, 1, 2, \cdots, 2^{2^{k'}}-1\}$,

22 is $2 \cdot 2^{2^{i+2j}} + 3 - 1$ in set $\{0, 1, 2, \cdots, 2^{2^{k'}} - 1\}$,

$\vdots \qquad \vdots \qquad \vdots$

$|f(i|j)||f(i|j)|$ is $|f(i|j)| \cdot 2^{2^{i+2j}} + (|f(i|j)| + 1) - 1$
in set $\{0, 1, 2, \cdots, 2^{2^{k'}} - 1\}$,

$\vdots \qquad \vdots \qquad \vdots$

$(2^{2^{i+2j}} - 1)(2^{2^{i+2j}} - 1)$ is $(2^{2^{i+2j}} - 1) \cdot 2^{2^{i+2j}} + [\,(2^{2^{i+2j}} - 1) + 1\,] - 1$
in set $\{0, 1, 2, \cdots, 2^{2^{k'}} - 1\}$,

[for the above, we could also use relation (1.25) with $\mu = \lambda$ and with convention (2.29) or (2.30) for the strings instead of (2.28)].

From the above, we see (for verification) that the last diagonal element $(2^{2^{i+2j}} - 1)(2^{2^{i+2j}} - 1)$ is indeed $2^{2^{i+2j+1}} - 1$, that is, the last element in the set of (3.15).

3 As for $|f(i|j)||f(i|j)|$, which is our main interest, we have just seen that

$$
\begin{gathered}
|f(i|j)||f(i|j)| = 2|f(i|j)| \equiv \\
|f(i|j)| \cdot 2^{2^{i+2j}} + (|f(i|j)| + 1) - 1 = \\
|f(i|j)| \cdot (2^{2^{i+2j}} + 1) \in \{0, 1, 2, \cdots, 2^{2^{k'}} - 1\},
\end{gathered} \tag{3.16}
$$

with $k' = k + 1 = i + 2j + 1$. This result compactly is presented in Table 3.2.

Table 3.2 How from $|f(i|j)|$ we obtain $2|f(i|j)|$.

For $k = i + 2j$ it is

$$
\begin{aligned}
|f(i|j)| = 1_k = 1_{i+2j} &\in \{0, 1, 2, \cdots, 2^{2^k} - 1\} \\
&= \{0, 1, 2, \cdots, 2^{2^{i+2j}} - 1\},
\end{aligned} \tag{3.17}
$$

and

$$
\begin{gathered}
2|f(i|j)| = |f(i|j)||f(i|j)| = \\
|f(i|j)| \cdot (2^{2^{i+2j}} + 1) \in \{0, 1, 2, \cdots, 2^{2^{k'}} - 1\},
\end{gathered} \tag{3.18}
$$

with $k' = k + 1 = i + 2j + 1$.

3.3 How from lines $|f(i|j)|$ or $2|f(i|j)|$ we obtain their multiple $4|f(i|j)|$.

1 Now from line $|f(i|j)|$ or from line $2|f(i|j)|$ we produce line $4|f(i|j)|$ which is line $|f(i|j)|$ taken four times or line $2|f(i|j)|$ taken twice. We work exactly as in the previous section when from line $|f(i|j)|$ we obtained its multiple line $2|f(i|j)|$. Since the procedure is same no extra descriptions are necessary and we present briefly the results.

Line $2|f(i|j)|$ is given in Table 3.2 and we consider it for k'. Also it is

$$2|f(i|j)| \in \{0, 1, 2, \cdots, 2^{2^{k'}} - 1\}. \tag{3.19}$$

Putting now $2|f(i|j)|$ and $4|f(i|j)|$ where in the procedure of Sec. 3.2 we have put $|f(i|j)|$ and $2|f(i|j)|$ respectively, we obtain what follows.

For $4|f(i|j)|$ it is

$$4|f(i|j)| = 2[\,2|f(i|j)|\,] = [\,2|f(i|j)|\,][\,2|f(i|j)|\,], \tag{3.20}$$

and this line now is considered for k'' with

$$k'' = k' + 1 = k + 2 = i + 2j + 2. \tag{3.21}$$

Thus

$$\begin{aligned} 4|f(i|j)| &\in \{0, 1, 2, \cdots, 2^{2^{k''}} - 1\} \\ &= \{0, 1, 2, \cdots, 2^{2^{k'+1}} - 1\} \\ &= \{0, 1, 2, \cdots, 2^{2^{i+2j+2}} - 1\}. \end{aligned} \tag{3.22}$$

In accordance with (3.12) the concatenation of $2|f(i|j)|$ with itself i.e. $4|f(i|j)|$ belongs to the set

$$\begin{array}{llllllll} \{ & \underline{00} & 01 & 02 & \cdots & 0(2|f|) & \cdots & 0(2^{2^{k'}} - 1) \\ & 10 & \underline{11} & 12 & \cdots & 1(2|f|) & \cdots & 1(2^{2^{k'}} - 1) \\ & 20 & 21 & \underline{22} & \cdots & 2(2|f|) & \cdots & 2(2^{2^{k'}} - 1) \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

$$
\begin{array}{ccccccc}
(2|f|)0 & (2|f|)1 & (2|f|)2 & \cdots & \underline{\underline{(2|f|)(2|f|)}} & \cdots & (2|f|)(2^{2^{k'}}-1) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(2^{2^{k'}}-1)0, & (2^{2^{k'}}-1)1, & (2^{2^{k'}}-1)2, & \cdots, & (2^{2^{k'}}-1)(2|f|), & \cdots, & \\
& & & & & \underline{(2^{2^{k'}}-1)(2^{2^{k'}}-1)} & \}
\end{array} \tag{3.23}
$$

The set (3.23), written in array-form, facilitates the enumeration of the new strings (resulting from the concatenation of the previous ones) and hence their renaming.

2 In accordance with (3.16) we obtain now for $(2|f(i|j)|)(2|f(i|j)|)$:

$$
\begin{aligned}
(2|f(i|j)|)(2|f(i|j)|) &= (2|f(i|j)|)\cdot 2^{2^{k'}} + [\,(2|f(i|j)|)+1\,]-1 \\
&= (2|f(i|j)|)\cdot[\,2^{2^{k'}}+1\,] = (2|f(i|j)|)\cdot[\,2^{2^{i+2j+1}}+1\,] \\
&\in \{0,1,2,\cdots,2^{2^{k''}}-1\}.
\end{aligned} \tag{3.24}
$$

Thus we produce directly the results in Table 3.3. Eq. (3.25) is produced by combining (3.20) and (3.24). Eq. (3.26) is produced by combining (3.25) and (3.18).

Table 3.3 How from $2|f(i|j)|$ or from $|f(i|j)|$ we obtain $4|f(i|j)|$.

$$
\begin{aligned}
4|f(i|j)| &= (2|f(i|j)|)\cdot[\,2^{2^{i+2j+1}}+1\,] \\
&\in \{0,1,2,\cdots,2^{2^{k''}}-1\},
\end{aligned} \tag{3.25}
$$

with $k'' = k'+1 = k+2 = i+2j+2$.

Equivalently

$$
\begin{aligned}
4|f(i|j)| &= |f(i|j)|\cdot(2^{2^{i+2j}}+1)\cdot(2^{2^{i+2j+1}}+1) \\
&\in \{0,1,2,\cdots,2^{2^{k''}}-1\},
\end{aligned} \tag{3.26}
$$

with $k'' = k'+1 = k+2 = i+2j+2$.

3.4 How from $|f(i|j)|$ or $4|f(i|j)|$ we obtain $8|f(i|j)|$.

1 Now working exactly as before we produce $8|f(i|j)|$ from $4|f(i|j)|$ or from $|f(i|j)|$. We present the results without comments.

It is

$$8|f(i|j)| = (4|f(i|j)|)(4|f(i|j)|). \tag{3.27}$$

The line $8|f(i|j)|$ is considered for k''' with

$$k''' = k'' + 1 = k' + 2 = k + 3 = i + 2j + 3, \tag{3.28}$$

thus

$$8|f(i|j)| \in \{0, 1, 2, \cdots, 2^{2^{k'''}} - 1\}. \tag{3.29}$$

In accordance with (3.12) or (3.23) we have the set

$$\begin{array}{lllllll} \{ \quad \underline{00} & 01 & 02 & \cdots & 0(4|f|) & \cdots & 0(2^{2^{k''}} - 1) \\ \quad \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (4|f|)0 & (4|f|)1 & (4|f|)2 & \cdots & \underline{\underline{(4|f|)(4|f|)}} & \cdots & (4|f|)(2^{2^{k''}} - 1) \\ \quad \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

$$(2^{2^{k''}} - 1)0,\ (2^{2^{k''}} - 1)1,\ (2^{2^{k''}} - 1)2,\ \cdots,\ (2^{2^{k''}} - 1)(4|f|),\ \cdots,$$

$$\underline{(2^{2^{k''}} - 1)(2^{2^{k''}} - 1)}\ \} \tag{3.30}$$

As in (3.16) and (3.24) we obtain now

$$(4|f(i|j)|)(4|f(i|j)|) = (4|f(i|j)|) \cdot 2^{2^{k''}} + [\, 4|f(i|j)| + 1\,] - 1. \tag{3.31}$$

2 In accordance with Table 3.3 we obtain Table 3.4.

Table 3.4 How from $4|f(i|j)|$ or from $|f(i|j)|$ we obtain $8|f(i|j)|$.

$$\begin{aligned} 8|f(i|j)| &= (4|f(i|j)|) \cdot [\, 2^{2^{k''}} + 1\,] \\ &\in \{0, 1, 2, \cdots, 2^{2^{k'''}} - 1\}, \end{aligned} \tag{3.32}$$

with $k''' = k'' + 1 = k' + 2 = k + 3 = i + 2j + 3$.

Equivalently

$$8|f(i|j)| = |f(i|j)| \cdot (2^{2^{i+2j}}+1) \cdot (2^{2^{i+2j+1}}+1) \cdot (2^{2^{i+2j+2}}+1)$$
$$\in \{0, 1, 2, \cdots, 2^{2^{k'''}}-1\}, \tag{3.33}$$

with $k''' = k'' + 1 = k' + 2 = k + 3 = i + 2j + 3$.

3.5 How we obtain $2^n|f(i|j)|$.

1 Now we can evidently generalize the previous results and produce line $2^n|f(i|j)|$. We write directly Table 3.5.

Table 3.5 How from $2^n|f(i|j)|$ we obtain $2^{n+1}|f(i|j)|$.

For k with $k = i + 2j$ it is

$$|f(i|j)| = 1_k \in \{0, 1, 2, \cdots, 2^{2^{i+2j}}-1\}. \tag{3.34}$$

For $k^{n \text{ accent-marks}}$ with $k^{n \text{ accent-marks}} = k^{n-1 \text{ accent-marks}} + 1 = \cdots = k'' + (n-2) = k' + (n-1) = k + n = i + 2j + n$ it is

$$2^n|f(i|j)| \in \{0, 1, 2, \cdots, 2^{2^{i+2j+n}}-1\}, \tag{3.35}$$

with $n = 0, 1, 2, \cdots$.

Also

$$2^{n+1}|f(i|j)| = (2^n|f(i|j)|) \cdot [\, 2^{2^{i+2j+n}}+1\,]$$
$$\in \{0, 1, 2, \cdots, 2^{2^{i+2j+n+1}}-1\}. \tag{3.36}$$

Eq. (3.36) contains a recursive relation by which $2^{n+1}|f(i|j)|$ is expressed with the help of $2^n|f(i|j)|$. By repetitive use of (3.36) we can express $2^n|f(i|j)|$ nonrecursively as function of $|f(i|j)|$. The result appears in Table 3.6.

Table 3.6 How from $|f(i|j)|$ we obtain $2^n|f(i|j)|$.

$$2^n|f(i|j)| =$$
$$|f(i|j)| \cdot (2^{2^{i+2j}} + 1) \cdot (2^{2^{i+2j+1}} + 1) \cdot (2^{2^{i+2j+2}} + 1) \cdots (2^{2^{i+2j+n-1}} + 1)$$
$$\in \{0, 1, 2, \cdots, 2^{2^{i+2j+n}} - 1\}, \tag{3.37}$$

with $n = 1, 2, \cdots$.

For $k = i + 2j$ it is

$$|f(i|j)| = 1_k \in \{0, 1, 2, \cdots, 2^{2^{i+2j}} - 1\}. \tag{3.38}$$

2 Thus the task posed at the beginning of Sec. 3.2, that is from $|f(i|j)|$ find $2|f(i|j)|$ and $4|f(i|j)|$ and $\cdots$ generally $2^n|f(i|j)|$, is accomplished in Tables 3.5 and 3.6 which contain the generalization of the results of the previous Tables 3.2 and 3.3 and 3.4.

Since $2^n|f(i|j)|$ is expressed as function of $|f(i|j)|$ which we choose to represent as a string of just one unit namely 1_k which is for the case of $k = i + 2j$, see Tables 2.15 and 2.16, we prefer to write 1_k separately to facilitate its frequent use. So string 1_k, which is analytically determined in Tables 2.13 and 2.14, is written again in Table 3.7 transferred directly from (2.121g).

Table 3.7 $|f(i|j)|$ and 1_k analytically written. 1_k is directly transferred from Table 2.13.

For $k = i + 2j$ it is

$$|f(i|j)| = 1_k = 1_{i+2j} \equiv$$
$$A_0 \cdot A_1 \cdot A_2 \cdots A_{i-2} \cdot \underbrace{A_i \cdot A_{i+1} \cdots A_{i+j-2}}_{\text{for } j=1 \text{ it does NOT exist!}}$$
$$\cdot Z \cdot \underbrace{A_{i+j} \cdot A_{i+j+1} \cdots A_{i+2j-2}}_{\text{for } j=1 \text{ it does NOT exist!}}, \tag{3.39}$$

with

$$A_m \equiv (2^{2^m} + 1), \tag{3.40}$$

and

$$Z \equiv [\, 2^{2^{i+2j-1}+2^{i+j-1}} + 2^{2^{i+2j-1}+2^{i-1}} + 2^{2^{i+j-1}+2^{i-1}}$$
$$+2^{2^{i+2j-1}} + 2^{2^{i+j-1}} + 2^{2^{i-1}}\,]. \qquad (3.41)$$

The basic nonrecursive equation of (3.37), taking into account (3.40), is written again in Table 3.8 more compactly.

Table 3.8 $2^n|f(i|j)|$, as given in (3.37), written with the help of (3.40).

$$2^n|f(i|j)| = |f(i|j)| \cdot A_{i+2j} \cdot A_{i+2j+1} \cdot A_{i+2j+2} \cdots A_{i+2j+n-1} \qquad (3.42)$$

3.6 Analysing further the formulae for $2^n|f(i|j)|$ as exponentials of two. Representing the formulae in the binary style.

1 Now we consider further $2^n|f(i|j)|$ and $|f(i|j)|$ with respect to the exponentials of two thus representing the formulae in a binary style. This binary style, from some aspects, facilitates the superposition of the lines as we have seen in Sec. 2.2 and especially in (2.26) and (2.36) where $l \circ m$ results directly. We remind also Note 3.1.

2 First, we note in Tables 3.7 and 3.8 that in the expressions of $|f(i|j)|$ and $2^n|f(i|j)|$ the products of A_m's may be naturally separated into the following parts:

$$\begin{array}{ll} \text{part (a):} & A_0 \cdot A_1 \cdot A_2 \cdots A_{i-2} \\ \text{part (b):} & A_i \cdot A_{i+1} \cdots A_{i+j-2} \\ \text{part (c):} & A_{i+j} \cdot A_{i+j+1} \cdots A_{i+2j-2} \\ \text{part (d):} & A_{i+2j} \cdot A_{i+2j+1} \cdots A_{i+2j+n-1} \\ \text{part (e):} & Z \end{array} \qquad (3.43)$$

In the formula of Table 3.7 part (e), which is Z, is written between part (b) and part (c). But for practical reasons in (3.43) we write Z separately at the end as part (e). This is permitted since here we have just products of integers.

We note, regarding (3.43), that:

between part (a) and part (b) factor A_{i-1} is missing,

between part (b) and part (c) factor A_{i+j-1} is missing,

between part (c) and part (d) factor A_{i+2j-1} is missing.

Thus, taking into account (3.39),(3.42),(3.43) and what we noted about "missing factors", Table 3.8 and (3.42) may be rewritten as Table 3.9 and (3.44).

Table 3.9 An equivalent version of Table 3.8.

$$2^n|f(i|j)| = A_0 \cdot A_1 \cdot A_2 \cdots A_{i+2j+n-1} \cdot Z \tag{3.44}$$

but in the product of the second part of Eq. (3.44) factors

$$A_{i-1}, \quad A_{i+j-1}, \quad A_{i+2j-1} \tag{3.45}$$

are missing.

So next we consider, for simplicity, the product

$$A_0 \cdot A_1 \cdot A_2 \cdots A_{i+2j+n-1}, \tag{3.46}$$

which has the general form

$$A_0 \cdot A_1 \cdot A_2 \cdots A_m, \tag{3.47}$$

and the missing factors (3.45) will be taken into account as a detail modifying slightly the results derived from the general forms (3.46) and (3.47). Also the special factor Z will be taken into account later as an addition modifying the results.

3 Now let us see examples for (3.47). First we consider the example $A_0 \cdot A_1 \cdot A_2$ and we have

$$A_0 \cdot A_1 \cdot A_2 = (2^{2^0} + 1) \cdot (2^{2^1} + 1) \cdot (2^{2^2} + 1) =$$

$$2^{2^0+2^1+2^2}+2^{2^0+2^1}+2^{2^0+2^2}+2^{2^1+2^2}+2^{2^0}+2^{2^1}+2^{2^2}+2^0=$$
$$2^7+2^3+2^5+2^6+2^1+2^2+2^4+2^0=$$
$$2^7+2^6+2^5+2^4+2^3+2^2+2^1+2^0. \tag{3.48}$$

Evidently the integer, represented as sum, in the last part of Eq. (3.48) may be written in the binary representation simply

$$11111111, \tag{3.49}$$

which is a sequence of eight digits 1.

Similarly as next example for (3.47) we consider $A_0 \cdot A_1 \cdot A_2 \cdot A_3$ and we have

$$A_0 \cdot A_1 \cdot A_2 \cdot A_3 =$$
$$(2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^2}+1)\cdot(2^{2^3}+1)=$$
$$2^{2^0+2^1+2^2+2^3}+2^{2^0+2^1+2^2}+2^{2^0+2^1+2^3}+2^{2^0+2^2+2^3}+2^{2^1+2^2+2^3}$$
$$+2^{2^0+2^1}+2^{2^0+2^2}+2^{2^0+2^3}+2^{2^1+2^2}+2^{2^1+2^3}+2^{2^2+2^3}$$
$$+2^{2^0}+2^{2^1}+2^{2^2}+2^{2^3}+2^0=$$
$$2^{15}+2^7+2^{11}+2^{13}+2^{14}+2^3+2^5+2^9+2^6+2^{10}+2^{12}+2^1+2^2+2^4+2^8+2^0=$$
$$2^{15}+2^{14}+\cdots+2^2+2^1+2^0, \tag{3.50}$$

and the integer, represented as sum, in the last part of Eq. (3.50) may be, in the binary representation, written

$$11\cdots111, \tag{3.51}$$

where there are sixteen digits 1.

4 For the general case (3.47) it is

$$A_0 \cdot A_1 \cdot A_2 \cdots A_m =$$
$$(2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^2}+1)\cdots(2^{2^m}+1). \tag{3.52}$$

We may analyse this product as we did in the examples.

It is possible, for simplicity, to denote the exponentials of the form

$$2^{2^0+2^1+2^2+\cdots+2^m} \text{ or } 2^{2^0+2^2+\cdots+2^m} \tag{3.53}$$

and so on, just as

$$012\cdots m \text{ or } 02\cdots m \tag{3.54}$$

and so on respectively. With this convention the product (3.52) may be analysed, according to (3.48) and (3.50), into the sum of the exponential terms

$$012\cdots m, \tag{3.55a}$$

and

$$012\cdots(m-1),\ 012\cdots(m-2)m,\ 012\cdots(m-3)(m-1)m,\cdots,$$
$$013\cdots m,\ 023\cdots m,\ 123\cdots m, \tag{3.55b}$$

and so on. The exponential terms (3.55) are just the first ones appearing in the sum. Totally the sum, into which the product (3.52) is analysed, contains the following kinds of exponential terms:

$$\begin{array}{llcc}
 & & \text{kind of term} & \text{missing numbers} \\
\binom{m+1}{0} & \text{terms} & 012\cdots m & 0 \\
\binom{m+1}{1} & \text{terms} & 012\cdots(m-1),\cdots & 1 \\
\binom{m+1}{2} & \text{terms} & 012\cdots(m-2),\cdots & 2 \\
\vdots & \vdots & \vdots & \vdots \\
\binom{m+1}{m-1} & \text{terms} & 01, 02, 03,\cdots & m-1 \\
\binom{m+1}{m} & \text{terms} & 0, 1, 2,\cdots, m & m \\
1 & \text{term} & 2^0 &
\end{array} \tag{3.56}$$

The meaning of (3.56) is clear: in the first row we say that in the sum, into which the product (3.52) is analysed, there are $\binom{m+1}{0}$ exponential terms i.e. one exponential term (since for the binomial coefficient it is $\binom{m+1}{0}=1$) of the kind $012\cdots m$ namely term $012\cdots m$ itself [which is the first exponential of (3.53)], and in the sequence $012\cdots m$ none of the numbers $0, 1, 2,\cdots, m$ is missing; similarly, in the second row we say that in the same sum there are $\binom{m+1}{1}$ exponential terms i.e. $m+1$ exponential terms (since for the binomial coefficient it is $\binom{m+1}{1}=m+1$) of the kind $012\cdots(m-1)$ etc. which analytically are the terms appearing in (3.55b) and these are the sequences of the form $012\cdots m$ in any one of which there is one missing number namely $m,(m-1),(m-2),\cdots,2,1,0$ respectively; similarly in the third row we have $\binom{m+1}{2}$ exponential terms of the form $012\cdots m$ where in each such term there are two missing numbers; $\cdots$ similarly in the next appearing row we have $\binom{m+1}{m-1}$ exponential terms of the form $012\cdots m$ with $m-1$ missing numbers in each such term, i.e., these terms analytically are $01, 02, 03, \cdots, (m-1)m$ and by them we mean $2^{2^0+2^1}, 2^{2^0+2^2}, 2^{2^0+2^3}, \cdots, 2^{2^{m-1}+2^m}$ respectively; similarly in the next row we have $\binom{m+1}{m}$ (and it is $\binom{m+1}{m} = m+1$) exponential terms of the form $012\cdots m$ with m missing numbers in each such term, i.e., these terms analytically are $0, 1, 2, \cdots, m$ and by them we mean $2^{2^0}, 2^{2^1}, 2^{2^2}, \cdots, 2^{2^m}$ respectively; finally in the next row of (3.56) we have one exponential term namely 2^0. All in (3.56) is clear if we have in mind the examples (3.48) and (3.50).

5 Now we cite some well-known results concerning binomial forms. It is

$$\binom{\mu}{\nu} = \frac{\mu!}{\nu!(\mu-\nu)!}. \tag{3.57}$$

Also we have the equality

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n, \tag{3.58}$$

for $n > 0$. The equality in (3.58) is obtained directly if in the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}, \tag{3.59}$$

we put $x = y = 1$. [(3.58) and (3.59) and other combinatorial results can be found in page 16 of *A. Papaioannou,Discrete Mathematics,Nat. Tech. University,Athens 2001*,(in Greek) and in page 68 of *D. Daskalopoulos,Typologion of Higher Mathematics,Nat. Tech. University,Athens 1979*(in Greek)].

Also we cite some results from Chapter 3 of the *Mathematical Diary 1993–1998* although they will be not used in the following. By *M.D. 1993–1998*,Ch. 3,Eq. (136) and Eq. (3.57) we obtain

$$\sum_m n = \binom{m+n}{m+1} = \frac{(m+n)!}{(m+1)!(n-1)!}$$

$$= \frac{n(n+1)(n+2)\cdots(n+m)}{(m+1)!}, \tag{3.60}$$

when $m \geqslant 0$, and $n \geqslant 1$. Also from *M.D. 1993–1998*,Ch. 3,Eq. (130) we have

$$\binom{p+0}{p} + \binom{p+1}{p} + \cdots + \binom{p+q}{p} = \binom{p+q+1}{p+1}. \tag{3.61}$$

6 According to (3.56) the total number of exponential terms, into which the product (3.52) is analysed, is

$$\binom{m+1}{0} + \binom{m+1}{1} + \binom{m+1}{2} + \cdots + \binom{m+1}{m-1} + \binom{m+1}{m} + 1. \tag{3.62}$$

Putting $n = m + 1$ into (3.58) we obtain

$$\binom{m+1}{0} + \binom{m+1}{1} + \binom{m+1}{2} + \cdots + \binom{m+1}{m+1} = 2^{m+1}. \tag{3.63}$$

From (3.63) we obtain

$$\binom{m+1}{0} + \binom{m+1}{1} + \binom{m+1}{2} + \cdots + \binom{m+1}{m} = 2^{m+1} - \binom{m+1}{m+1}. \tag{3.64}$$

Combining (3.62) with (3.64) and taking into account that $\binom{m+1}{m+1} = 1$ we have for the total number of exponential terms in (3.62) that

$$\binom{m+1}{0} + \binom{m+1}{1} + \binom{m+1}{2} + \cdots + \binom{m+1}{m} + 1$$

$$= 2^{m+1} - \binom{m+1}{m+1} + 1 = 2^{m+1} - 1 + 1 = 2^{m+1}. \tag{3.65}$$

So having totally 2^{m+1} exponential terms in the sum into which the product (3.52) is analysed, we may write the equality in Table 3.10.

Table 3.10 The product (3.52).

$$A_0 \cdot A_1 \cdot A_2 \cdots A_m =$$

$$2^{2^{m+1}-1} + 2^{2^{m+1}-2} + \cdots + 2^2 + 2^1 + 2^0 = \sum_{k=0}^{2^{m+1}-1} 2^k \qquad (3.66)$$

(We can prove easily (3.66) by generalizing the procedure appearing in the examples (3.48) and (3.50)).

7 Let us consider the product $A_{n+0} \cdot A_{n+1} \cdots A_{n+m}$. It is

$$A_{n+0} \cdot A_{n+1} \cdots A_{n+m} =$$

$$(2^{2^{n+0}} + 1) \cdot (2^{2^{n+1}} + 1) \cdots (2^{2^{n+m}} + 1) =$$

$$2^{2^{n+0}+2^{n+1}+\cdots+2^{n+m}} + \cdots = 2^{2^n \cdot (2^0+2^1+\cdots+2^m)} + \cdots =$$

$$2^{2^n \cdot (2^{m+1}-1)} + 2^{2^n \cdot (2^{m+1}-2)} + \cdots + 2^{2^n \cdot 2} + 2^{2^n \cdot 1} + 2^{2^n \cdot 0}. \qquad (3.67)$$

We can see the truth of (3.67) with the help of an example, say for $n = 5$ and $m = 3$, having in mind (3.50).

So we write Table 3.11.

Table 3.11 The product (3.67).

$$A_{n+0} \cdot A_{n+1} \cdots A_{n+m} =$$

$$2^{(2^{m+1}-1)\cdot 2^n} + 2^{(2^{m+1}-2)\cdot 2^n} + \cdots + 2^{2\cdot 2^n} + 2^{1\cdot 2^n} + 2^{0\cdot 2^n} = \sum_{k=0}^{2^{m+1}-1} 2^{k\cdot 2^n} \qquad (3.68)$$

We have produced (3.68) wishing to use it for determining the products appearing in (3.43). Indeed, applying (3.68) to these products we obtain the results presented in Table 3.12.

Table 3.12 The products in (3.43).

$$A_0 \cdot A_1 \cdot A_2 \cdots A_{i-2} =$$

$$2^{(2^{i-1}-1)} + 2^{(2^{i-1}-2)} + \cdots + 2^2 + 2^1 + 2^0 = \sum_{k=0}^{2^{i-1}-1} 2^{k\cdot 2^0} \tag{3.69a}$$

$$A_i \cdot A_{i+1} \cdots A_{i+j-2} = A_{\underline{i}+0} \cdot A_{\underline{i}+1} \cdots A_{\underline{i}+j-2} =$$

$$2^{(2^{j-1}-1)\cdot 2^i} + 2^{(2^{j-1}-2)\cdot 2^i} + \cdots + 2^{2\cdot 2^i} + 2^{1\cdot 2^i} + 2^{0\cdot 2^i} = \sum_{k=0}^{2^{j-1}-1} 2^{k\cdot 2^i} \tag{3.69b}$$

$$A_{i+j} \cdot A_{i+j+1} \cdots A_{i+2j-2} = A_{\underline{i+j}+0} \cdot A_{\underline{i+j}+1} \cdots A_{\underline{i+j}+j-2} =$$

$$2^{(2^{j-1}-1)\cdot 2^{i+j}} + 2^{(2^{j-1}-2)\cdot 2^{i+j}} + \cdots + 2^{2\cdot 2^{i+j}} + 2^{1\cdot 2^{i+j}} + 2^{0\cdot 2^{i+j}} = \sum_{k=0}^{2^{j-1}-1} 2^{k\cdot 2^{i+j}} \tag{3.69c}$$

$$A_{i+2j} \cdot A_{i+2j+1} \cdots A_{i+2j+n-1} = A_{\underline{i+2j}+0} \cdot A_{\underline{i+2j}+1} \cdots A_{\underline{i+2j}+n-1} =$$

$$2^{(2^{n}-1)\cdot 2^{i+2j}} + 2^{(2^{n}-2)\cdot 2^{i+2j}} + \cdots + 2^{2\cdot 2^{i+2j}} + 2^{1\cdot 2^{i+2j}} + 2^{0\cdot 2^{i+2j}} = \sum_{k=0}^{2^{n}-1} 2^{k\cdot 2^{i+2j}} \tag{3.69d}$$

and part

$$Z \tag{3.69e}$$

The above equalities result directly from (3.68) if we put in it: $n = 0$, $m = i - 2$ for (3.69a); $n = i$, $m = j - 2$ for (3.69b); $n = i + j$, $m = j - 2$ for (3.69c); $n = i + 2j$, $m = n - 1$ for (3.69d). In (3.69) we underline the parts of the indices of A's that correspond to symbol n of (3.68). We see that (3.69a)–(3.69e) correspond respectively to parts (a)–(e) of (3.43).

According to Tables 3.7 and 3.8 the product of the four subproducts in (3.69a)–(3.69d) and of Z in (3.69e) gives $2^n|f(i|j)|$.

8 Now we consider again the general case of the product $A_0 \cdot A_1 \cdot A_2 \cdots A_m$ appearing in (3.47),(3.52), and (3.66). We shall see what happens in this product if we omit a random factor A_i.

Omitting A_i the product (3.47) becomes

$$A_0 \cdot A_1 \cdot A_2 \cdots A_{i-1} \cdot A_{i+1} \cdots A_m, \tag{3.70}$$

and we may equivalently write it

$$A_0 \cdot A_1 \cdot A_2 \cdots A_{i-1} \underbrace{\cdot A_i}_{\text{omitted}} \cdot A_{i+1} \cdots A_m, \tag{3.71}$$

which analytically is

$$(2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^2}+1)\cdots(2^{2^{i-1}}+1)\underbrace{\cdot(2^{2^i}+1)}_{\text{omitted}}\cdot(2^{2^{i+1}}+1)\cdots(2^{2^m}+1). \tag{3.72}$$

It is clear that in the exponential terms, into which the product (3.72) is analysed [see also (3.53)–(3.56)], the term $2^{\,i}$ is missing from the exponents of 2.

9 Let us represent the exponential terms, into which the product (3.47) or (3.71) is analysed, just by writing the exponents into a specific type of binary form. As example we consider this product for $m = 3$ i.e. $A_0 \cdot A_1 \cdot A_2 \cdot A_3$, see (3.50). The exponential terms, into which this product is analysed, appear in (3.50) and analytically they are

$$\begin{gathered}
2^{2^0+2^1+2^2+2^3}, \\
2^{2^0+2^1+2^2}, 2^{2^0+2^1+2^3}, 2^{2^0+2^2+2^3}, 2^{2^1+2^2+2^3}, \\
2^{2^0+2^1}, 2^{2^0+2^2}, 2^{2^0+2^3}, 2^{2^1+2^2}, 2^{2^1+2^3}, 2^{2^2+2^3}, \\
2^{2^0}, 2^{2^1}, 2^{2^2}, 2^{2^3}, \\
2^0.
\end{gathered} \tag{3.73}$$

Supposing that, in (3.73), in any exponent of 2 there exist all the exponentials $2^0, 2^1, 2^2, 2^3$ each having a factor 0 or 1 in front of it, and instead of these exponentials writing only these factors without "+" (for example, in exponential $2^{2^0+2^1+2^3}$ the exponent $2^0+2^1+2^3$ is considered as $1 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + 1 \cdot 2^3$ and we write it simply as 1101), we represent (3.73) equivalently as

$$\begin{gathered}
2^{1111}, \\
2^{1110}, 2^{1101}, 2^{1011}, 2^{0111},
\end{gathered}$$

$$2^{1100}, 2^{1010}, 2^{1001}, 2^{0110}, 2^{0101}, 2^{0011},$$
$$2^{1000}, 2^{0100}, 2^{0010}, 2^{0001},$$
$$2^{0000}. \tag{3.74}$$

The digits 0 and 1 here are cited in reverse order compared to the familiar binary representation of integers. For example, by 1100 we mean $1 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + 0 \cdot 2^3$ whereas in the familiar binary representation we should write it $0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$ hence 0011 or just 11.

For simplicity we may write only the exponents of the exponentials in (3.74), and in these we may replace digits 0 and 1 by letters b and a respectively. Thus, for example, exponential 2^{1010} may be written simply 1010 or equivalently $abab$. With this convention (3.74) may be written

$$aaaa,$$
$$aaab, aaba, abaa, baaa,$$
$$aabb, abab, abba, baab, baba, bbaa,$$
$$abbb, babb, bbab, bbba,$$
$$bbbb. \tag{3.75}$$

So the product (3.47) for $m = 3$, see (3.50), is analysed into the sum of the exponential terms (3.75) which are represented, in brief form, as strings.

10 Now let us see what happens if in the product $A_0 \cdot A_1 \cdot A_2 \cdot A_3$ we omit e.g. A_2. This means, in accordance with (3.70)–(3.72), to consider the product

$$A_0 \cdot A_1 \cdot A_3, \tag{3.76}$$

which is

$$A_0 \cdot A_1 \underbrace{\cdot A_2}_{\text{omitted}} \cdot A_3, \tag{3.77}$$

which is

$$(2^{2^0} + 1) \cdot (2^{2^1} + 1) \underbrace{\cdot (2^{2^2} + 1)}_{\text{omitted}} \cdot (2^{2^3} + 1). \tag{3.78}$$

Since factor $(2^{2^2} + 1)$ is cancelled in the product (3.78), it is clear that in the exponents of the exponentials of 2, into the sum of which

(3.78) is analysed, the terms 2^2 are cancelled too, see (3.73). This means that in the sum of the four terms $2^0, 2^1, 2^2, 2^3$ term 2^2 will always be with factor 0 in front of it, so in the exponents in (3.74) the third digit becomes always 0. Hence in (3.75) the third letter of any string becomes b.

Thus omitting A_2 in product (3.77) and considering product (3.76) has as result, in (3.75), to put letter b in the third site of any string. This means that the corresponding to (3.75) for the product (3.76) is

$$aaba, aabb, abba, baba, abbb, babb, bbba, bbbb. \tag{3.79}$$

(After transforming the sixteen strings of (3.75) each resulting string is same with another string of the transformed (3.75). Because of the structure of the product (3.78) it is clear that each such string in (3.79) is taken just once. So in (3.79) there are eight strings.) The strings (3.79) correspond respectively to the exponential terms

$$2^{2^0+2^1+2^3}, 2^{2^0+2^1}, 2^{2^0+2^3}, 2^{2^1+2^3}, 2^{2^0}, 2^{2^1}, 2^{2^3}, 2^0, \tag{3.80}$$

into the sum of which the product (3.76) is analysed.

Note 3.13 From the product (3.78) we may obtain directly the strings (3.75) or (3.79) by the following trick: in any string each letter is taken from (corresponds to) a specific factor of the product (3.78); so the letter is put a if it is obtained from the first term (the term with the exponential) of the corresponding factor of (3.78), and b if it is obtained from the second term (the term with integer one) of the corresponding factor of (3.78).

As an example, let us consider string $abab$ of (3.75). This string evidently corresponds to the exponential term $2^{2^0+2^2}$, see (3.73), that is obtained if, in the product (3.78), we select: the first term (hence we write a) of the first factor, i.e., term 2^{2^0}; the second term (hence we write b) of the second factor, i.e., term 1; the first term (hence we write a) of the third factor, i.e., term 2^{2^2}; the second term (hence we write b) of the fourth factor, i.e., term 1.

As another example, let us consider string $abba$ of (3.79). This string corresponds to the exponential term $2^{2^0+2^3}$, see (3.80), that is obtained if, in the product (3.78), we select: the first term (hence we write a) of the first factor, i.e., term 2^{2^0}; the second term (hence we write b) of the second factor, i.e., term 1; the second term (hence we write b) of the third factor, i.e., term 1 [since the third factor of (3.78) is missing we consider it as 1, hence from the two terms of the missing factor we consider always term 1 and this means that we always write letter b in the third site of the strings (3.79)]; the first term (hence we write a) of the fourth factor, i.e., term 2^{2^3}. □

11 Thus for the product $A_0 \cdot A_1 \cdot A_2 \cdots A_m$ [this is (3.47)] we may define the following auxiliary set:

$$A \equiv \{\text{every string of } m+1 \text{ letters, with each letter being } a \text{ or } b\}. \tag{3.81}$$

For example, for $m = 3$ set A contains all the strings (3.75).

If we delete (omit) factor A_i in the product $A_0 \cdot A_1 \cdot A_2 \cdots A_m$ [hence considering the product (3.70)] we may define the auxiliary set:

$$A_i \equiv \{\text{every string of } m+1 \text{ letters, with each letter being } a \text{ or } b\text{, and in the site that corresponds to index } i \text{ (we call it } i\text{-site and it is the } (i+1)\text{th site of the string) having always letter } b\}. \tag{3.82}$$

For example, for $m = 3$ and $i = 2$, set A_2 contains all the strings (3.79). [Of course we can easily, from the context, distinct when A_i represents a factor in the product (3.47) and when it represents set (3.82).]

Clearly it is

$$A_i \subset A. \tag{3.83}$$

Similarly, we can see that if in the product (3.47) we omit more than one factors, for example if we omit factors A_i and A_j or if we omit factors A_i and A_j and A_k, we obtain analogous subsets of set A. Thus from the first example we obtain subset $A_{i,j}$ and from the second example we obtain subset $A_{i,j,k}$. Subset $A_{i,j}$ is the set of all strings of $m+1$ letters with every letter being a or b and in the i-site and j-site, which are the $(i+1)$th and $(j+1)$th site of the string respectively, having letter b. Subset $A_{i,j,k}$ is the set of all strings of $m+1$ letters with every letter being a or b and in the i-site and j-site and k-site, which are the $(i+1)$th and $(j+1)$th and $(k+1)$th site of the string respectively, having letter b.

12 We have, so far, developed tools for representing better the products in (3.42),(3.44),(3.46) — see also (3.43). Applying the previous results to the product (3.46), which is

$$A_0 \cdot A_1 \cdot A_2 \cdots A_{i+2j+n-1}, \tag{3.84}$$

we have that $m = i+2j+n-1$ and that the auxiliary set A contains all the strings with number of letters $m+1$, where $m+1 = i+2j+n$, and with each letter being a or b.

Dropping out (omitting) factors

$$A_{i-1}, A_{i+j-1}, A_{i+2j-1} \tag{3.85}$$

from the product (3.84) we obtain the set

$$A_{i-1,i+j-1,i+2j-1} \tag{3.86}$$

which contains all the strings with $m+1$ ($m+1 = i+2j+n$) letters a or b, whose $(i-1)-, (i+j-1)-, (i+2j-1)-$ sites [i.e. the (i)th,$(i+j)$th,$(i+2j)$th sites of each string] have letter b.

13 Now, having performed detailed analysis of the product (3.84), let us take into account also factor Z which means let us see how the complete product

$$A_0 \cdot A_1 \cdot A_2 \cdots A_{i+2j+n-1} \cdot Z, \tag{3.87}$$

where the three factors $A_{i-1}, A_{i+j-1}, A_{i+2j-1}$ are not included [i.e. the product in (3.44)], is represented in the familiar style with the exponentials of 2 and the strings of letters a and b. As we have already noted in the last sentence of the Subsection 2 of the present Section 3.6, considering the extra special factor Z the results for product (3.46) will be properly modified. This consideration of the factor Z will take place in the following through various steps.

At this point we notice that Z, which was introduced in (2.119) [see also (3.41)], contains just the exponents $i-1, i+j-1, i+2j-1$ in the exponentials of 2 which are exactly the indices of the *dropped out factors* $A_{i-1}, A_{i+j-1}, A_{i+2j-1}$! Using for simplicity letters α, β, γ instead of these indices, that is, writing

$$(\alpha, \beta, \gamma) \equiv (i-1, i+j-1, i+2j-1), \tag{3.88}$$

we have

$$Z = (2^{2^\alpha + 2^\beta} + 2^{2^\alpha + 2^\gamma} + 2^{2^\beta + 2^\gamma} + 2^{2^\alpha} + 2^{2^\beta} + 2^{2^\gamma}). \tag{3.89}$$

14 So numbers α, β, γ in the product (3.87) *do not appear* as indices of the factors A but they are *the (upper) exponents* in the special factor Z.

Now some examples will help to clarify the above. We write formula (3.42), in which $|f(i|j)|$ is taken from (3.39), for various values of i, j, n. Thus we obtain

$$
\begin{aligned}
2^1|f(1|1)| = \;& (2^{2^2+2^1} + 2^{2^2+2^0} + 2^{2^1+2^0} + 2^{2^2} + 2^{2^1} + 2^{2^0}) \\
& \cdot(2^{2^3}+1) \\
2^2|f(1|1)| = \;& (2^{2^2+2^1} + 2^{2^2+2^0} + 2^{2^1+2^0} + 2^{2^2} + 2^{2^1} + 2^{2^0}) \\
& \cdot(2^{2^3}+1)\cdot(2^{2^4}+1) \\
2^3|f(1|1)| = \;& (2^{2^2+2^1} + 2^{2^2+2^0} + 2^{2^1+2^0} + 2^{2^2} + 2^{2^1} + 2^{2^0}) \\
& \cdot(2^{2^3}+1)\cdot(2^{2^4}+1)\cdot(2^{2^5}+1) \\
\vdots \qquad & \qquad\qquad \vdots
\end{aligned}
$$

$$
\begin{aligned}
2^1|f(1|2)| = \;& (2^{2^1}+1)\cdot(2^{2^4+2^2} + 2^{2^4+2^0} + 2^{2^2+2^0} + 2^{2^4} + 2^{2^2} + 2^{2^0})\cdot(2^{2^3}+1) \\
& \cdot(2^{2^5}+1) \\
2^2|f(1|2)| = \;& (2^{2^1}+1)\cdot(2^{2^4+2^2} + 2^{2^4+2^0} + 2^{2^2+2^0} + 2^{2^4} + 2^{2^2} + 2^{2^0})\cdot(2^{2^3}+1) \\
& \cdot(2^{2^5}+1)\cdot(2^{2^6}+1) \\
2^3|f(1|2)| = \;& (2^{2^1}+1)\cdot(2^{2^4+2^2} + 2^{2^4+2^0} + 2^{2^2+2^0} + 2^{2^4} + 2^{2^2} + 2^{2^0})\cdot(2^{2^3}+1) \\
& \cdot(2^{2^5}+1)\cdot(2^{2^6}+1)\cdot(2^{2^7}+1) \\
\vdots \qquad & \qquad\qquad \vdots
\end{aligned}
$$

$$
\begin{aligned}
2^1|f(1|3)| = \;& (2^{2^1}+1)\cdot(2^{2^2}+1)\cdot(2^{2^6+2^3} + 2^{2^6+2^0} + 2^{2^3+2^0} + 2^{2^6} + 2^{2^3} + 2^{2^0}) \\
& \cdot(2^{2^4}+1)\cdot(2^{2^5}+1)\cdot(2^{2^7}+1) \\
2^2|f(1|3)| = \;& (2^{2^1}+1)\cdot(2^{2^2}+1)\cdot(2^{2^6+2^3} + 2^{2^6+2^0} + 2^{2^3+2^0} + 2^{2^6} + 2^{2^3} + 2^{2^0}) \\
& \cdot(2^{2^4}+1)\cdot(2^{2^5}+1)\cdot(2^{2^7}+1)\cdot(2^{2^8}+1) \\
2^3|f(1|3)| = \;& (2^{2^1}+1)\cdot(2^{2^2}+1)\cdot(2^{2^6+2^3} + 2^{2^6+2^0} + 2^{2^3+2^0} + 2^{2^6} + 2^{2^3} + 2^{2^0}) \\
& \cdot(2^{2^4}+1)\cdot(2^{2^5}+1)\cdot(2^{2^7}+1)\cdot(2^{2^8}+1)\cdot(2^{2^9}+1) \\
\vdots \qquad & \qquad\qquad \vdots
\end{aligned}
$$

$$
\ddots \qquad\qquad \ddots
$$

$$
\begin{aligned}
2^1|f(2|1)| = \;& (2^{2^0}+1)\cdot(2^{2^3+2^2} + 2^{2^3+2^1} + 2^{2^2+2^1} + 2^{2^3} + 2^{2^2} + 2^{2^1}) \\
& \cdot(2^{2^4}+1) \\
2^2|f(2|1)| = \;& (2^{2^0}+1)\cdot(2^{2^3+2^2} + 2^{2^3+2^1} + 2^{2^2+2^1} + 2^{2^3} + 2^{2^2} + 2^{2^1}) \\
& \cdot(2^{2^4}+1)\cdot(2^{2^5}+1) \\
2^3|f(2|1)| = \;& (2^{2^0}+1)\cdot(2^{2^3+2^2} + 2^{2^3+2^1} + 2^{2^2+2^1} + 2^{2^3} + 2^{2^2} + 2^{2^1}) \\
& \cdot(2^{2^4}+1)\cdot(2^{2^5}+1)\cdot(2^{2^6}+1) \\
\vdots \qquad & \qquad\qquad \vdots
\end{aligned}
$$

$$\begin{aligned}
2^1|f(2|2)| = &\ (2^{2^0}+1)\cdot(2^{2^2}+1)\cdot(2^{2^5+2^3}+2^{2^5+2^1}+2^{2^3+2^1}+2^{2^5}+2^{2^3}+2^{2^1})\\
&\cdot(2^{2^4}+1)\cdot(2^{2^6}+1)\\
2^2|f(2|2)| = &\ (2^{2^0}+1)\cdot(2^{2^2}+1)\cdot(2^{2^5+2^3}+2^{2^5+2^1}+2^{2^3+2^1}+2^{2^5}+2^{2^3}+2^{2^1})\\
&\cdot(2^{2^4}+1)\cdot(2^{2^6}+1)\cdot(2^{2^7}+1)\\
2^3|f(2|2)| = &\ (2^{2^0}+1)\cdot(2^{2^2}+1)\cdot(2^{2^5+2^3}+2^{2^5+2^1}+2^{2^3+2^1}+2^{2^5}+2^{2^3}+2^{2^1})\\
&\cdot(2^{2^4}+1)\cdot(2^{2^6}+1)\cdot(2^{2^7}+1)\cdot(2^{2^8}+1)\\
\vdots\qquad & \qquad\qquad\qquad\vdots
\end{aligned}$$

$$\begin{aligned}
2^1|f(2|3)| = &\ (2^{2^0}+1)\cdot(2^{2^2}+1)\cdot(2^{2^3}+1)\\
&\cdot(2^{2^7+2^4}+2^{2^7+2^1}+2^{2^4+2^1}+2^{2^7}+2^{2^4}+2^{2^1})\\
&\cdot(2^{2^5}+1)\cdot(2^{2^6}+1)\cdot(2^{2^8}+1)\\
2^2|f(2|3)| = &\ (2^{2^0}+1)\cdot(2^{2^2}+1)\cdot(2^{2^3}+1)\\
&\cdot(2^{2^7+2^4}+2^{2^7+2^1}+2^{2^4+2^1}+2^{2^7}+2^{2^4}+2^{2^1})\\
&\cdot(2^{2^5}+1)\cdot(2^{2^6}+1)\cdot(2^{2^8}+1)\cdot(2^{2^9}+1)\\
2^3|f(2|3)| = &\ (2^{2^0}+1)\cdot(2^{2^2}+1)\cdot(2^{2^3}+1)\\
&\cdot(2^{2^7+2^4}+2^{2^7+2^1}+2^{2^4+2^1}+2^{2^7}+2^{2^4}+2^{2^1})\\
&\cdot(2^{2^5}+1)\cdot(2^{2^6}+1)\cdot(2^{2^8}+1)\cdot(2^{2^9}+1)\cdot(2^{2^{10}}+1)\\
\vdots\qquad & \qquad\qquad\qquad\vdots
\end{aligned}$$

$$\ddots \qquad\qquad\qquad\qquad \ddots$$

$$\begin{aligned}
2^1|f(3|1)| = &\ (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^4+2^3}+2^{2^4+2^2}+2^{2^3+2^2}+2^{2^4}+2^{2^3}+2^{2^2})\\
&\cdot(2^{2^5}+1)\\
2^2|f(3|1)| = &\ (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^4+2^3}+2^{2^4+2^2}+2^{2^3+2^2}+2^{2^4}+2^{2^3}+2^{2^2})\\
&\cdot(2^{2^5}+1)\cdot(2^{2^6}+1)\\
2^3|f(3|1)| = &\ (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^4+2^3}+2^{2^4+2^2}+2^{2^3+2^2}+2^{2^4}+2^{2^3}+2^{2^2})\\
&\cdot(2^{2^5}+1)\cdot(2^{2^6}+1)\cdot(2^{2^7}+1)\\
\vdots\qquad & \qquad\qquad\qquad\vdots
\end{aligned}$$

$$\begin{aligned}
2^1|f(3|2)| = &\ (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^3}+1)\\
&\cdot(2^{2^6+2^4}+2^{2^6+2^2}+2^{2^4+2^2}+2^{2^6}+2^{2^4}+2^{2^2})\\
&\cdot(2^{2^5}+1)\cdot(2^{2^7}+1)\\
2^2|f(3|2)| = &\ (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^3}+1)\\
&\cdot(2^{2^6+2^4}+2^{2^6+2^2}+2^{2^4+2^2}+2^{2^6}+2^{2^4}+2^{2^2})\\
&\cdot(2^{2^5}+1)\cdot(2^{2^7}+1)\cdot(2^{2^8}+1)\\
2^3|f(3|2)| = &\ (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^3}+1)\\
&\cdot(2^{2^6+2^4}+2^{2^6+2^2}+2^{2^4+2^2}+2^{2^6}+2^{2^4}+2^{2^2})\\
&\cdot(2^{2^5}+1)\cdot(2^{2^7}+1)\cdot(2^{2^8}+1)\cdot(2^{2^9}+1)\\
\vdots\qquad & \qquad\qquad\qquad\vdots
\end{aligned}$$

$$\begin{aligned}
2^1|f(3|3)| = \; & (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^3}+1)\cdot(2^{2^4}+1) \\
& \cdot(2^{2^8+2^5}+2^{2^8+2^2}+2^{2^5+2^2}+2^{2^8}+2^{2^5}+2^{2^2}) \\
& \cdot(2^{2^6}+1)\cdot(2^{2^7}+1)\cdot(2^{2^9}+1) \\
2^2|f(3|3)| = \; & (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^3}+1)\cdot(2^{2^4}+1) \\
& \cdot(2^{2^8+2^5}+2^{2^8+2^2}+2^{2^5+2^2}+2^{2^8}+2^{2^5}+2^{2^2}) \\
& \cdot(2^{2^6}+1)\cdot(2^{2^7}+1)\cdot(2^{2^9}+1)\cdot(2^{2^{10}}+1) \\
2^3|f(3|3)| = \; & (2^{2^0}+1)\cdot(2^{2^1}+1)\cdot(2^{2^3}+1)\cdot(2^{2^4}+1) \\
& \cdot(2^{2^8+2^5}+2^{2^8+2^2}+2^{2^5+2^2}+2^{2^8}+2^{2^5}+2^{2^2}) \\
& \cdot(2^{2^6}+1)\cdot(2^{2^7}+1)\cdot(2^{2^9}+1)\cdot(2^{2^{10}}+1)\cdot(2^{2^{11}}+1) \\
\vdots \quad & \qquad\qquad\qquad\qquad \vdots \\
& \ddots \qquad\qquad\qquad \ddots \\
& \cdot \qquad\qquad \cdot \qquad\qquad \cdot
\end{aligned} \tag{3.90}$$

15 Wishing to write (3.90) in a form more compact, a form displaying only the distribution of the exponents in the exponents, we use the following trickery. We write the equality of $2^1|f(1|1)|$, appearing in (3.90), as follows:

$$2^1|f(1|1)| = (21, 20, 10, 2, 1, 0)|3. \tag{3.91}$$

Here only the upper exponents are shown. For example 21, in the parenthesis in (3.91), denotes the upper exponents of $2^{2^2+2^1}$ whereas 1 denotes the upper exponent of 2^{2^1}; integer 3, after the vertical line, denotes the upper exponent in $(2^{2^3}+1)$. Similarly the equalities for $2^2|f(1|1)|$ and $2^3|f(1|1)|$, in (3.90), are written respectively as follows:

$$2^2|f(1|1)| = (21, 20, 10, 2, 1, 0)|3, 4 \tag{3.92}$$

and

$$2^3|f(1|1)| = (21, 20, 10, 2, 1, 0)|3, 4, 5 \tag{3.93}$$

We can write the three equalities (3.91),(3.92), and (3.93), that correspond to the equalities for $2^1|f(1|1)|, 2^2|f(1|1)|$, and $2^3|f(1|1)|$ in (3.90), together as follows [we include dots that denote the corresponding vertical dots in (3.90)]:

$$2^n|f(1|1)| = (21, 20, 10, 2, 1, 0)|3, 4, 5, \cdots \tag{3.94}$$

for $n = 1, 2, 3, \cdots$. The vertical line after the parenthesis and before the numbers 3,4,5 in (3.94), as well as in (3.91)–(3.93), separates the part of this expression that is common for all values of n (this is the part on the left of the line) from the part of the expression that depends on the specific value of n (this is the part on the right of the line). In the parenthesis in (3.94) there are numbers 0,1,2. We may denote these three numbers by the initial number 0 and the step 1, that is, first number 0, second number 0+1 which is 1, third number 1+1 which is 2. The initial number 0 and the step 1 are written together as (0,1) and this is put at the right side of (3.94) which becomes:

$$2^n|f(1|1)| = (21, 20, 10, 2, 1, 0)|3, 4, 5, \cdots \qquad (0, 1) \tag{3.95}$$

for $n = 1, 2, 3, \cdots$.

Writing all the equalities of (3.90) in the style of (3.95), which was written for the first three of them, the whole (3.90) takes the form:

$$2^n|f(1|1)| = (21, 20, 10, 2, 1, 0)|3, 4, 5, \cdots \qquad (0, 1)$$

$$2^n|f(1|2)| = 1(42, 40, 20, 4, 2, 0)3|5, 6, 7, \cdots \qquad (0, 2)$$

$$2^n|f(1|3)| = 1, 2(63, 60, 30, 6, 3, 0)4, 5|7, 8, 9, \cdots \qquad (0, 3)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$2^n|f(2|1)| = 0(32, 31, 21, 3, 2, 1)|4, 5, 6, \cdots \qquad (1, 1)$$

$$2^n|f(2|2)| = 0, 2(53, 51, 31, 5, 3, 1)4|6, 7, 8, \cdots \qquad (1, 2)$$

$$2^n|f(2|3)| = 0, 2, 3(74, 71, 41, 7, 4, 1)5, 6|8, 9, 10, \cdots \qquad (1, 3)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$2^n|f(3|1)| = 0, 1(43, 42, 32, 4, 3, 2)|5, 6, 7, \cdots \qquad (2, 1)$$

$$2^n|f(3|2)| = 0, 1, 3(64, 62, 42, 6, 4, 2)5|7, 8, 9, \cdots \qquad (2, 2)$$

$$2^n|f(3|3)| = 0, 1, 3, 4(85, 82, 52, 8, 5, 2)6, 7|9, 10, 11, \cdots \qquad (2, 3)$$

$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ \ddots & \ddots & \ddots \end{array} \tag{3.96}$$

for $n = 1, 2, 3, \cdots$. As we see from the column at the right side of (3.96) [the column with the pairs displaying the "initial number" and the "step"] to any $2^n|f(i|j)|$ there corresponds pair $(i-1, j)$. We also see, in (3.96), that for any $2^n|f(i|j)|$ we have the sequence of numbers $0, 1, 2, \cdots, i-1+2j+n$; in any such sequence the numbers $i-1, i-1+j, i-1+2j$ and only these are displayed inside the parenthesis [that corresponds to the specific $2^n|f(i|j)|$] separately of the other numbers of the sequence which are displayed exclusively outside the parenthesis. We can see all these in Tables 3.7 and 3.8.

3.7 A detailed description of how the superposition of the lines $|f(i|j)|$ takes place with the help of the previously developed machinery.

1 Now we are going to see in detail how the previously developed machinery and formalism can be used for the superposition of the lines $|f(i|j)|$. This will be done with the help of an example. Specifically we are going to superpose $2^2|f(1|1)|$ with $|f(1|2)|$ thus obtaining $|f(1|1,2)|$.

From (3.90) and Table 3.7 we have

$$2^2|f(1|1)| = (2^{2^2+2^1}+2^{2^2+2^0}+2^{2^1+2^0}+2^{2^2}+2^{2^1}+2^{2^0})\cdot(2^{2^3}+1)\cdot(2^{2^4}+1), \tag{3.97}$$

and

$$|f(1|2)| = (2^{2^1}+1)\cdot(2^{2^4+2^2}+2^{2^4+2^0}+2^{2^2+2^0}+2^{2^4}+2^{2^2}+2^{2^0})\cdot(2^{2^3}+1), \tag{3.98}$$

where, in both, the maximal exponent is 4.

According to the style of (3.44) and (3.87) we may write (3.97) and (3.98) as follows:

$$2^2|f(1|1)| = A_3 \cdot A_4 \cdot Z, \tag{3.99}$$

and

$$|f(1|2)| = A_1 \cdot A_3 \cdot Z', \tag{3.100}$$

with

$$
\begin{aligned}
A_1 &= (2^{2^1} + 1), \\
A_3 &= (2^{2^3} + 1), \\
A_4 &= (2^{2^4} + 1), \\
Z &= (2^{2^2+2^1} + 2^{2^2+2^0} + 2^{2^1+2^0} + 2^{2^2} + 2^{2^1} + 2^{2^0}), \\
Z' &= (2^{2^4+2^2} + 2^{2^4+2^0} + 2^{2^2+2^0} + 2^{2^4} + 2^{2^2} + 2^{2^0}).
\end{aligned}
\tag{3.101}
$$

2 We may equivalently write (3.99) and (3.100) as follows:

$$2^2|f(1|1)| = \underbrace{A_0}_{\text{omitted}} \cdot \underbrace{A_1}_{\text{omitted}} \cdot \underbrace{A_2}_{\text{omitted}} \cdot A_3 \cdot A_4 \cdot Z, \tag{3.102}$$

and

$$|f(1|2)| = \underbrace{A_0}_{\text{omitted}} \cdot A_1 \cdot \underbrace{A_2}_{\text{omitted}} \cdot A_3 \cdot \underbrace{A_4}_{\text{omitted}} \cdot Z'. \tag{3.103}$$

Let us consider only the products of the A's, in (3.102) and (3.103), without omitted factors which means let us consider the product

$$A_0 \cdot A_1 \cdot A_2 \cdot A_3 \cdot A_4 \tag{3.104}$$

for both $2^2|f(1|1)|$ and $|f(1|2)|$. According to (3.81) all the exponential terms, existing in product (3.104), can be represented as the set [set A in (3.81)] of all strings

$$* * * * * \tag{3.105}$$

with each $*$ being a or b; these strings, evidently, are all the strings of five letters [$m = 4$ hence $m + 1 = 5$, see (3.81)] with each letter being a or b.

If we take into account the fact that, in product (3.104), there are omitted factors [which are A_0, A_1, A_2 for $2^2|f(1|1)|$ and A_0, A_2, A_4 for $|f(1|2)|$, see (3.102) and (3.103) respectively] then from this product only part

$$A_3 \cdot A_4 \tag{3.106}$$

and part

$$A_1 \cdot A_3 \tag{3.107}$$

remains respectively for $2^2|f(1|1)|$ and $|f(1|2)|$, see (3.99) and (3.100). As we know, see (3.82) and the related discussion, the set of all the exponential terms, existing in product (3.106), can be represented as the set of all strings

$$bbb**\qquad(3.108)$$

with $*$ being a or b, and the corresponding set for the product (3.107) is the set of all strings

$$b*b*b\qquad(3.109)$$

with $*$ again being a or b. [According to the discussion just after (3.83) we can denote the set of the strings in (3.108) as $A_{0,1,2}$ and the set of the strings in (3.109) as $A_{0,2,4}$ having in mind the omitted factors in (3.104) for each case respectively. The indices of these omitted factors, as we know (see also (3.85) and (3.86)), are the triples $(i-1, i+j-1, i+2j-1)$ which are equal to (0,1,2) for $2^2|f(1|1)|$ since $i=j=1$, and to (0,2,4) for $|f(1|2)|$ since $i=1$ and $j=2$.].

3 Now we are taking into account also the factor of type Z by multiplying (3.106) and (3.107) by Z and Z' respectively thus obtaining the second part of (3.99) and (3.100).

If, regarding $2^2|f(1|1)|$, we wish to find all the exponential terms existing in the second part of (3.99), which is equal to the second part of (3.97), we just combine (i.e., we multiply) any string of the type (3.108) [which represents an exponential term in the product (3.106)] with any exponential term existing in factor Z with this factor appearing analytically in (3.97) and in (3.101). We express all these combinations as follows:

$$\begin{array}{c}\text{For } 2^2|f(1|1)| \text{ we combine}\\[1ex]\begin{array}{cccc}\text{any} & bbb** & \text{with} & 2^{2^2+2^1}\\ ” & & ” & 2^{2^2+2^0}\\ ” & & ” & 2^{2^1+2^0}\\ ” & & ” & 2^{2^2}\\ ” & & ” & 2^{2^1}\\ ” & & ” & 2^{2^0}\end{array}\end{array}\qquad(3.110)$$

In the right column of (3.110) we see all the exponential terms existing in Z. Using, for representing the exponential terms of Z, the notation introduced in (3.91)–(3.96) [in which any such term is denoted by just the most upper exponents, for example, we denote term $2^{2^2+2^1}$ just by 21] we denote these exponential terms just by

$$(21, 20, 10, 2, 1, 0) \tag{3.111}$$

and better in set form as

$$\{21, 20, 10, 2, 1, 0\}. \tag{3.112}$$

Thus we may express the content of (3.110) simply as:

$$\text{“for } 2^2|f(1|1)| \text{ any string } bbb{*}{*} \text{ is combined with any of } \{21, 20, 10, 2, 1, 0\}\text{”} \tag{3.113}$$

Working for $|f(1|2)|$ in exactly same way as for $2^2|f(1|1)|$ we finally say that:

$$\text{“for } |f(1|2)| \text{ any string } b*b*b \text{ is combined with any of } \{42, 40, 20, 4, 2, 0\}\text{”} \tag{3.114}$$

4 We can write, for $2^2|f(1|1)|$, the following sets:

$$\begin{aligned} A \equiv\ & \{\text{exponential terms corresponding to any string } bbb{*}{*} \text{ with } * \text{ being } a \text{ or } b\}, \\ B \equiv\ & \{\text{exponential terms corresponding to } 21, 20, 10, 2, 1, 0\}. \end{aligned} \tag{3.115}$$

Similarly we can write, for $|f(1|2)|$, the sets:

$$\begin{aligned} C \equiv\ & \{\text{exponential terms corresponding to any string } b*b*b \text{ with } * \text{ being } a \text{ or } b\}, \\ D \equiv\ & \{\text{exponential terms corresponding to } 42, 40, 20, 4, 2, 0\}. \end{aligned} \tag{3.116}$$

The above sets may be expressed analytically and without words as follows:

for $2^2|f(1|1)|$

$$\begin{aligned} A &= \{bbbaa, bbbab, bbbba, bbbbb\}, \\ B &= \{21, 20, 10, 2, 1, 0\}, \end{aligned} \tag{3.117}$$

and for $|f(1|2)|$

$$\begin{aligned} C &= \{babab, babbb, bbbab, bbbbb\}, \\ D &= \{42, 40, 20, 4, 2, 0\}. \end{aligned} \tag{3.118}$$

If we write explicitly the exponential terms, which in (3.117) and (3.118) are represented by strings of letters or integers according to the established conventions [see the examples in (3.73)–(3.75),(3.79),(3.80) as well as in (3.90)–(3.96)], sets A, B, C, D take the form:

$$\begin{aligned} A &= \{2^{2^3+2^4}, 2^{2^3}, 2^{2^4}, 2^0\}, \\ B &= \{2^{2^2+2^1}, 2^{2^2+2^0}, 2^{2^1+2^0}, 2^{2^2}, 2^{2^1}, 2^{2^0}\}, \\ C &= \{2^{2^1+2^3}, 2^{2^1}, 2^{2^3}, 2^0\}, \\ D &= \{2^{2^4+2^2}, 2^{2^4+2^0}, 2^{2^2+2^0}, 2^{2^4}, 2^{2^2}, 2^{2^0}\}. \end{aligned} \tag{3.119}$$

5 Let us define by $A \otimes B$ the set of all possible products between the members of A and the members of B [as in cartesian product $A \times B$ but here, i.e. in $A \otimes B$, instead of pairs of the elements of A and B we have *products* of these elements]. Also we denote, for simplicity, any exponential by just its most upper exponents. For example, taking element $2^{2^3+2^4}$ of A and element 2^{2^2} of B their product is $2^{2^2+2^3+2^4}$ and we denote it just as 234.

Similar things hold for sets C and D and their product $C \otimes D$.

Taking into account (3.119), and using the convention of representing the exponentials by their most upper exponents, we write analytically, in (3.120) and (3.121), the products $A \otimes B$ and $C \otimes D$; the common elements of these two products are underlined.

$$\begin{aligned} A \otimes B = \; & \{\underline{1234}, 0234, \underline{0134}, \underline{234}, \underline{134}, \underline{034}, \\ & \underline{123}, \underline{023}, \underline{013}, \underline{23}, 13, \underline{03}, \\ & \underline{124}, 024, \underline{014}, \underline{24}, \underline{14}, \underline{04}, \\ & \underline{12}, \underline{02}, \underline{01}, \underline{2}, 1, \underline{0}\}, \end{aligned} \tag{3.120}$$

$$\begin{aligned} C \otimes D = \; & \{\underline{1234}, \underline{0134}, 0123, \underline{134}, \underline{123}, \underline{013}, \\ & \underline{124}, \underline{014}, 012, \underline{14}, \underline{12}, \underline{01}, \\ & \underline{234}, \underline{034}, \underline{023}, 34, \underline{23}, \underline{03}, \\ & \underline{24}, \underline{04}, \underline{02}, 4, \underline{2}, \underline{0}\}. \end{aligned} \tag{3.121}$$

The intersection of set $A\otimes B$ and set $C\otimes D$, which consists of the common (underlined) elements of these two product-sets displayed in (3.120) and (3.121), evidently is

$$(A\otimes B)\cap(C\otimes D) = \{1234, 0134, 234, 134, 034, 123, 023, 013, 124, 014, 23, 03, 24, 14, 04, 12, 02, 01, 2, 0\}. \quad (3.122)$$

So all the exponential terms existing in $2^2|f(1|1)|$ and in $|f(1|2)|$ are the exponential terms existing respectively in $A\otimes B$ and in $C\otimes D$. *The superposition of* $2^2|f(1|1)|$ *and* $|f(1|2)|$ *is determined from the intersection* $(A\otimes B)\cap(C\otimes D)$ and discussion on this is the content of the following important section.

3.8 Three important comments.

Now we are going to present some important things mainly through examples.

Comment 1: The procedure by which, in Sec. 1.1, the strings are renamed again and again is just the procedure by which the binary numbers, i.e. the integers written in the binary system, are produced.

1 Let us see Comment 1. Here for the strings l we use the convention (2.29) instead of (2.28) because convention (2.29) is more appropriate for the binary representation of the associated to the strings integers. According to (1.1) we have strings 0 and 1 which trivially are renamed 0 and 1 [we follow convention (2.29) for the strings resulting from the renaming] respectively which in the binary system are the numbers zero and one. Next, according to (1.4), the strings

$$00,\ \ 01,\ \ 10,\ \ 11 \quad (3.123)$$

are, following convention (2.29), renamed

$$0,\ \ 1,\ \ 2,\ \ 3 \quad (3.124)$$

respectively. Each of the strings (3.123) is just the corresponding integer (3.124) written in the binary system. We insist on using always two digits for each of the binary numbers (3.123): for example we write 01 for representing integer $0\cdot 2^1+1\cdot 2^0$, which is integer one, and not just 1 (representing $1 \cdot 2^0$) which would be more appropriate for the binary system.

2 Next we have (1.11) which, written with convention (2.29) for the new names of the strings, takes the form

$$\begin{array}{cccc} 0000 & 0001 & 0010 & 0011 \\ 0 & 1 & 2 & 3 \\ \\ 0100 & 0101 & 0110 & 0111 \\ 4 & 5 & 6 & 7 \\ \\ 1000 & 1001 & 1010 & 1011 \\ 8 & 9 & 10 & 11 \\ \\ 1100 & 1101 & 1110 & 1111 \\ 12 & 13 & 14 & 15 \end{array} \tag{3.125}$$

It is clear that strings $0000, 0001, \cdots, 1111$ in (3.125), which are renamed $0, 1, \cdots, 15$ respectively, are just the representation of integers $0, 1, \cdots, 15$ in the binary system (always with four digits and for this purpose we add superfluous digits 0 at the left side of the strings when necessary). For example, in (3.125) string 0101 is indeed the representation of integer 5 in the binary system with a superfluous 0 on the left: 0101 means $0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$ which is equal to 5.

We can easily see that the "mechanism" by which strings (3.123) and (3.125) and so on are produced in Sec. 1.1 is same with the "mechanism" by which the binary numbers are produced. We start with the trivial strings and binary numbers 0 and 1. We produce the next two-digit strings or binary numbers by adding in front of 0 and 1 again 0 and 1 in all possible combinations. So we obtain in lexicographic order

$$00 \;\; 01 \;\; 10 \;\; 11 \tag{3.126}$$

The strings or binary numbers (3.126) in the decimal representation are integers 0,1,2,3.

For producing the next three-digit binary numbers we add in front of numbers (3.126) again 0 and 1 in all possible combinations obtaining in lexicographic order

$$\begin{array}{cccc} 000 & 001 & 010 & 011 \\ 100 & 101 & 110 & 111 \end{array} \tag{3.127}$$

which in decimal system are the integers 0,1,2,3,4,5,6,7. Then we may, in the same way, produce the next four-digit binary numbers by adding in front of the numbers (3.127) again 0 and 1 in all possible combinations obtaining in lexicographic order

$$\begin{array}{cccc} 0000 & 0001 & 0010 & 0011 \\ 0100 & 0101 & 0110 & 0111 \\ 1000 & 1001 & 1010 & 1011 \\ 1100 & 1101 & 1110 & 1111 \end{array} \tag{3.128}$$

which in the decimal system are integers $0, 1, 2, \cdots, 15$.

So in (3.127) and (3.128) we have seen the "mechanism" through which from the two-digit binary numbers (3.126) we produce the four-digit binary numbers (3.128). Let us see now the "mechanism" by which in Sec. 1.1 from the two-digit strings (3.126) we produce the four-digit strings (3.125). This "mechanism" is just that in front of the two-digit strings (3.126) we add again the same strings in all possible combinations thus obtaining directly (i.e. without passing through the intermediary step with the three-digit strings) the four-digit strings, in lexicographic order,

$$\begin{array}{cccc} 0000 & 0001 & 0010 & 0011 \\ 0100 & 0101 & 0110 & 0111 \\ 1000 & 1001 & 1010 & 1011 \\ 1100 & 1101 & 1110 & 1111 \end{array} \tag{3.129}$$

which are exactly the four-digit binary numbers (3.128).

Thus in this example Comment 1 is verified: the procedure (the "mechanism") by which from the two-digit strings (3.126) we obtain, according to Sec. 1.1, the four-digit strings (3.129) is the same with the procedure ("mechanism") by which from the two-digit binary numbers (3.126) we obtain [through the intermediary step with the three-digit binary numbers (3.127)] the four-digit binary numbers (3.128) which are (3.129). Since the same procedure, in Sec. 1.1, produces the strings with more and more digits it is clear that *these strings and their ordering are just the corresponding binary numbers with their natural ordering.*

3 Let us present the above procedure more formally. Regarding the formation of the binary numbers we can write (0 1) to denote the elementary binary numbers 0 and 1. Then to produce the two-digit binary numbers we write

$$(0\ \ 1) \cdot (0\ \ 1) = 0 \cdot (0\ \ 1)\ \ 1 \cdot (0\ \ 1) = 00\ \ 01\ \ 10\ \ 11 \tag{3.130}$$

and the meaning is clear: we just produce the binary numbers (3.126). We better write (3.130) omitting dots and empty spaces for simplicity. So (3.130) becomes

$$(01)(01) = 0(01)1(01) = 00\ \ 01\ \ 10\ \ 11 \tag{3.131}$$

Continuing in the same way we produce the three-digit binary numbers, in the style of (3.131), as follows:

$$(01)(01)(01) = (01)((01)(01)) = (01)(00\ \ 01\ \ 10\ \ 11) =$$

$$\begin{array}{cccc} 000 & 001 & 010 & 011 \\ 100 & 101 & 110 & 111 \end{array} \tag{3.132}$$

And of course the final part of (3.132) is (3.127). Similarly we produce the four-digit binary numbers:

$$(01)(01)(01)(01) = (01)((01)(01)(01)) =$$

$$(01)\Big(\begin{array}{cccc} 000 & 001 & 010 & 011 \\ 100 & 101 & 110 & 111 \end{array}\Big) =$$

$$\begin{array}{cccc} 0000 & 0001 & 0010 & 0011 \\ 0100 & 0101 & 0110 & 0111 \\ 1000 & 1001 & 1010 & 1011 \\ 1100 & 1101 & 1110 & 1111 \end{array} \tag{3.133}$$

the final part of (3.133) being (3.128).

We may, similarly, continue producing the binary numbers with more digits.

For simplicity instead of (01) and (01)(01) and so on we may write respectively 1(01) and 2(01) and so on. Analytically we have

$$\begin{array}{rl} (01) = & 1(01) \\ (01)(01) = & 2(01) \\ (01)(01)(01) = & 3(01) \\ \vdots & \vdots \end{array} \tag{3.134}$$

4 Now as we have formalized, in (3.130)–(3.134), the production of the binary numbers we may similarly formalize the production of the strings as described in Sec. 1.1. Specifically we write the elementary strings 0 and 1 as (01). Then we produce the two-digit strings (we use the same notation as above for the binary numbers) as follows:

$$(01)(01) = 00 \;\; 01 \;\; 10 \;\; 11 \tag{3.135}$$

Then we produce the four-digit strings as follows:

$$((01)(01))((01)(01)) = (00 \;\; 01 \;\; 10 \;\; 11)(00 \;\; 01 \;\; 10 \;\; 11) =$$

$$\begin{matrix} 0000 & 0001 & 0010 & 0011 \\ 0100 & 0101 & 0110 & 0111 \\ 1000 & 1001 & 1010 & 1011 \\ 1100 & 1101 & 1110 & 1111 \end{matrix} \tag{3.136}$$

Similarly we may continue and produce the eight-digit strings:

$$(((01)(01))((01)(01)))(((01)(01))((01)(01))) = \cdots \tag{3.137}$$

And so on for the strings with more digits.

Applying our well-known convention with the asterisks at the end of a string, which means that we double the string [for example $a*$ is aa, $a**$ is $aa*$ which is $aaaa$, and so on — see also (2.95)–(2.97) and (2.90),(2.91) and many other formulae], we may write the first parts of (3.135)–(3.137) briefly as follows:

$$\begin{aligned} (01) \equiv&\;\; (01) &&\equiv (01)0* \\ (01)(01) \equiv&\;\; (01)* &&\equiv (01)1* \\ ((01)(01))((01)(01)) \equiv&\;\; (01)** &&\equiv (01)2* \end{aligned} \tag{3.138}$$

the first part of (3.137) being similarly $(01)***$ or $(01)3*$. In the general case we have n asterisks, i.e., $(01)**\cdots*$ which is $(01)n*$.

Since the last parts of (3.131) and (3.135) are same their first parts are also equal. The same holds also for (3.133) and (3.136), as well as for the next similar examples that may be presented. So we have

$$\begin{aligned} (01) &= (01) \\ (01)(01) &= (01)(01) \\ (01)(01)(01)(01) &= ((01)(01))((01)(01)) \\ &\vdots \end{aligned} \tag{3.139}$$

Using the abbreviations adopted in (3.134) and (3.138) regarding respectively the first and second parts of (3.139) we may write compactly (3.139) as follows:

$$\begin{array}{rcl} 1(01) & = & (01) \\ 2(01) & = & (01)* \\ 4(01) & = & (01)** \\ 8(01) & = & (01)*** \\ \vdots & & \vdots \end{array} \tag{3.140}$$

or equivalently

$$\begin{array}{rcl} 2^0(01) & = & (01)0* \\ 2^1(01) & = & (01)1* \\ 2^2(01) & = & (01)2* \\ 2^3(01) & = & (01)3* \\ \vdots & & \vdots \\ 2^n(01) & = & (01)n* \end{array} \tag{3.141}$$

Thus we may, instead of (3.140) and (3.141), write equivalently just

$$2^n(01) = (01)n*, \tag{3.142}$$

with $n = 0, 1, 2, \cdots$.

It is clear that (3.142) is valid from the way it has been obtained. Of course we can easily prove formally (3.142) by the method of mathematical induction but this is not necessary here.

So the discussion on Comment 1 is complete.

Comment 2: The line $|f(i|j)|$ or $2P^{\,i}_{\,j}$ usually is represented as a sequence of digits 0 and 1, see the examples (2.38)–(2.40). This sequence may be considered as an integer written in the binary system. This integer, represented in the decimal system, is just the integer 1_k representing the sequence (the string) for $k = i + 2j$, see Tables 2.13–2.16 as well as Tables (3.5)–(3.7). For the strings always we accept the convention (2.29).

5 Let us see Comment 2 through a simple example. We have, see (2.1) or (2.38a),

$$|f(1|1)| = 01111110 \tag{3.143}$$

According to the procedure described in Sec. 1.1 the above string is written in the convention (2.29) and for $k = 0$. According to the procedure of Sec. 1.1, and having in mind Comment 1, we see that when we rename the string (3.143) to come to the case of $k = 1$ what essentially happens is: string (3.143) is separated into its four consistuent pairs 01,11,11,10; considering each such pair as an integer written in the binary system, these four integers in the decimal system are 1,3,3,2 (for example, 11 is $1 \cdot 2^1 + 1 \cdot 2^0$ which is 3); then string (3.143), for $k = 1$, is renamed 1332, that is, we have

$$|f(1|1)| = 1332 \tag{3.144}$$

following always convention (2.29) [for the renaming see also (3.123) and (3.124)].

Proceeding further to case $k = 2$ and renaming again string (3.143), according to Sec. 1.1, what essentially we do is: we separate string (3.143) into its two consistuent parts 0111 and 1110; considering each such part as an integer written in the binary system, these two integers in the decimal system are 7 and 14 (for example, 0111 is $0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$ which is 7); then string (3.143), for $k = 2$, is renamed 7 $\underline{14}$, that is, we have

$$|f(1|1)| = 7\ \underline{14} \tag{3.145}$$

following always convention (2.29) [for the renaming see also (3.125)]. As in the past, see (2.8)–(2.9), we underline 14 to emphasize that it is the integer fourteen and not concatenation of 1 and 4.

Finally, renaming string (3.143) for case $k = 3$, according to Sec. 1.1, practically we do the following: we take string (3.143) as a whole; we consider this eight-digit string as an integer written in the binary system; this integer in the decimal system is 126 [since 01111110 is

$$0 \cdot 2^7 + 1 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0, \tag{3.146}$$

which is equal to 126]; so renaming string (3.143), finally for $k = 3$, we obtain

$$|f(1|1)| = \underline{126} \tag{3.147}$$

or just

$$|f(1|1)| = 126 \tag{3.148}$$

since 126 is clearly a decimal integer and not a concatenation. Always convention (2.29) holds.

6 As another brief example, let us consider (2.11) which is (2.39a). We have

$$|f(1|2)| = 0101111101011111111101011111010 \qquad (3.149)$$

This string may be considered as an integer written in the binary system. This means that the integer, omitting the zero terms, is

$$2^{30} + 2^{28} + 2^{27} + 2^{26} + 2^{25} + 2^{24} + 2^{22} + 2^{20} + 2^{19} + 2^{18} + 2^{17}$$

$$+2^{16}+2^{15}+2^{14}+2^{13}+2^{12}+2^{11}+2^{9}+2^{7}+2^{6}+2^{5}+2^{4}+2^{3}+2^{1} \quad (3.150)$$

which is equal to the sum of integers

$$1073741824 + 268435456 + 134217728 + 67108864 + 33554432$$

$$+16777216 + 4194304 + 1048576 + 524288 + 262144 + 131072$$

$$+65536 + 32768 + 16384 + 8192 + 4096 + 2048 + 512 + 128$$

$$+64 + 32 + 16 + 8 + 2 \qquad (3.151)$$

which is equal to the integer

$$1600125690 \qquad (3.152)$$

written in the decimal system. Since the binary integer of the second part of (3.149) is, in the decimal system, the integer (3.152) we may write

$$|f(1|2)| = 1600125690 \qquad (3.153)$$

This result, following convention (2.29), is in agreement with (2.14).

7 As verification of the results (3.148) and (3.153) through the formulae of Tables (2.13)–(2.16) and of Tables (3.5)–(3.7), we have indeed from these tables, see especially (3.39)–(3.41), that: for $|f(1|1)|$, since $i = j = 1$ and $k = i + 2j = 3$, it is

$$|f(1|1)| = 1_3 = Z = (2^{2^2+2^1} + 2^{2^2+2^0} + 2^{2^1+2^0} + 2^{2^2} + 2^{2^1} + 2^{2^0}) =$$

$$2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 2^1 = 64 + 32 + 16 + 8 + 4 + 2 = 126, \quad (3.154)$$

where clearly the sum $2^6+2^5+\cdots+2^1$ is the binary integer in (3.143); similarly for $|f(1|2)|$, since $i = 1$ and $j = 2$ and $k = i + 2j = 5$, it is

$$|f(1|2)| = 1_5 = A_1 \cdot Z \cdot A_3 =$$

$$
\begin{gathered}
(2^{2^1}+1)\cdot(2^{2^4+2^2}+2^{2^4+2^0}+2^{2^2+2^0}+2^{2^4}+2^{2^2}+2^{2^0})\cdot(2^{2^3}+1)= \\
(2^2+1)\cdot(2^{20}+2^{17}+2^5+2^{16}+2^4+2^1)\cdot(2^8+1)= \\
(2^2+1)\cdot(2^8+1)\cdot(2^{20}+2^{17}+2^5+2^{16}+2^4+2^1)= \\
(2^{10}+2^8+2^2+1)\cdot(2^{20}+2^{17}+2^{16}+2^5+2^4+2^1)= \\
\\
2^{10}\cdot(2^{20}+2^{17}+2^{16}+2^5+2^4+2^1)+ \\
2^{8}\cdot(2^{20}+2^{17}+2^{16}+2^5+2^4+2^1)+ \\
2^{2}\cdot(2^{20}+2^{17}+2^{16}+2^5+2^4+2^1)+ \\
1\cdot(2^{20}+2^{17}+2^{16}+2^5+2^4+2^1)= \\
\\
(2^{30}+2^{27}+2^{26}+2^{15}+2^{14}+2^{11})+ \\
(2^{28}+2^{25}+2^{24}+2^{13}+2^{12}+2^{9})+ \\
(2^{22}+2^{19}+2^{18}+2^{7}+2^{6}+2^{3})+ \\
(2^{20}+2^{17}+2^{16}+2^5+2^4+2^1)= \\
\\
2^{30}+2^{28}+2^{27}+2^{26}+2^{25}+2^{24}+ \\
2^{22}+2^{20}+2^{19}+2^{18}+2^{17}+2^{16}+ \\
2^{15}+2^{14}+2^{13}+2^{12}+2^{11}+2^{9}+ \\
2^{7}+2^{6}+2^{5}+2^{4}+2^{3}+2^{1},
\end{gathered}
\tag{3.155}
$$

and the last part of (3.155) is equal to (3.150) which is equal to (3.151) which is equal to (3.152) [and the last part of (3.155), or (3.150), is clearly the binary integer in the second part of (3.149)].

So the results (3.154) and (3.155) verify respectively the results (3.148) and (3.153). These examples clarify adequately Comment 2 and we do not insist on providing strict proofs.

8 A string $|f(i|j)|$ is composed of digits 0 or 1. The total number of digits 1 contained in the string has been denoted as $nz(|f(i|j)|)$

and more generally for any string as $nz(string)$, see *Mathematical Diary 1993–1998*,Chapter 7,§1 and especially Remark 1 and Eq. (12) [nz stands for "nonzero points" which correspond to digits 1 of the string]. Taking into account Comment 2 we can see directly that: *If we have a string* $|f(i|j)|$ *written elementarily for* $k = 0$*, i.e. as a sequence of the elementary digits 0 or 1, we can represent it, according to Tables 2.13–2.16 and to Tables 3.5–3.7, for* $k = i + 2j$ *just as a decimal integer* 1_k *(it may generally be denoted as* l*) belonging, see convention (2.29), to set* $\{0, 1, 2, \cdots, 2^{2^k} - 1\}$*. The initial (i.e., for* $k = 0$*) elementary string* $|f(i|j)|$ *is the integer* 1_k *written as a binary integer. The total number of digits 1 in this elementary string, which we call* $nz(|f(i|j)|)$*, is evidently the total number of the exponential terms of two in the sum of such terms into which the decimal integer* 1_k *is analysed, see the representation of* 1_k *in Tables 2.13–2.16 and in Tables 3.5–3.7.* In reverse order we may say that: *Any string represented by an integer* l*, such that* $l \in \{0, 1, 2, \cdots, 2^{2^k} - 1\}$*, (in this procedure* l *is* 1_k *for* $k = i + 2j$*) has "numerical value"* $nz(l)$ *which is the total number of digits 1 in the sequence of digits 0 and 1 resulting when we write the decimal integer* l *in its binary form. This total number of digits 1 is just the total number of exponential terms of two in the binary form of integer* l*. We mean of course the nonzero exponential terms, i.e. the terms with coefficient 1, in the sum of the exponential terms (each term having coefficient 0 or 1) which expresses analytically the binary integer, see for example the binary integer in (3.143) and its expression as the sum (3.146).*

We proceed now to the important Comment 3.

Comment 3: Suppose we have two strings (lines) l *and* m *of the familiar in this work kind and we wish to obtain the "numerical value"* $nz(l \circ m)$ *of the superposition* $l \circ m$*. We consider* l *and* m *as binary forms of integers. Each binary form of an integer, which is a sequence of digits 0 and 1, corresponds to a sum of exponential terms of two. So* l*, as a binary integer, corresponds to a set of exponential terms of two which is the set of those such terms that in the corresponding sum have coefficient 1 (the nonzero exponential terms of two in the sum). In the same way* m *corresponds to a similar set. We can easily see, just from the definition of the superposition* $l \circ m$ *of the lines* l *and* m*, that the numerical value* $nz(l \circ m)$ *is equal to the* *total number of the exponential terms that the above two sets (i.e., the sets that correspond to l and to m respectively) have in common.*

9 We clarify Comment 3 by an example. Let us consider as l and m respectively the lines $2^2|f(1|1)|$ and $|f(1|2)|$. We wish to write their superposition $l \circ m$. The line $2^2|f(1|1)|$ is four times the line $|f(1|1)$ which analytically appears in (3.143). So it is

$$2^2|f(1|1)| = 01111110011111100111111001111110. \tag{3.156}$$

The line $|f(1|2)|$ appears analytically in (3.149). Each of the lines in (3.156) and in (3.149) has 32 digits which are superposed site-by-site as we can see analytically in (20) of Chapter 7 of the *Mathematical Diary 1993–1998*. We cite again this superposition denoting $2^2|f(1|1)|$ as l and $|f(1|2)|$ as m respectively:

$$\begin{array}{c|l} l & 01111110011111100111111001111110 \\ m & 01011111010111111111101011111010 \\ \hline l \circ m & 01011110010111100111101001111010 \end{array} \tag{3.157}$$

It is known that for the site-by-site superposition of the lines when a site 0 is superposed with a site 0 or 1 the resulting site is 0 and when a site 1 is superposed with a site 0 or 1 the resulting site is 0 or 1 respectively. That is, the only case the resulting site to be 1 is when the superposed sites *are both* 1.

Let us consider now lines l and m and $l \circ m$, which are strings of digits 0 and 1, as binary integers. Then the binary integer in the first line of (3.157), which is the line l [see also (3.156)], analytically is

$$\begin{gathered} 0 \cdot 2^{31} + 1 \cdot 2^{30} + 1 \cdot 2^{29} + 1 \cdot 2^{28} + 1 \cdot 2^{27} + 1 \cdot 2^{26} + 1 \cdot 2^{25} + 0 \cdot 2^{24} + \\ 0 \cdot 2^{23} + 1 \cdot 2^{22} + 1 \cdot 2^{21} + 1 \cdot 2^{20} + 1 \cdot 2^{19} + 1 \cdot 2^{18} + 1 \cdot 2^{17} + 0 \cdot 2^{16} + \\ 0 \cdot 2^{15} + 1 \cdot 2^{14} + 1 \cdot 2^{13} + 1 \cdot 2^{12} + 1 \cdot 2^{11} + 1 \cdot 2^{10} + 1 \cdot 2^{9} + 0 \cdot 2^{8} + \\ 0 \cdot 2^{7} + 1 \cdot 2^{6} + 1 \cdot 2^{5} + 1 \cdot 2^{4} + 1 \cdot 2^{3} + 1 \cdot 2^{2} + 1 \cdot 2^{1} + 0 \cdot 2^{0}. \end{gathered} \tag{3.158}$$

In (3.158) the nonzero exponential terms are those with coefficient 1 and their sum produces the integer since the contribution of the other terms is zero.

Similarly the binary integer in the second line of (3.157), which is the line m [see also (3.149)], analytically is

$$\begin{gathered}0\cdot 2^{31}+1\cdot 2^{30}+0\cdot 2^{29}+1\cdot 2^{28}+1\cdot 2^{27}+1\cdot 2^{26}+1\cdot 2^{25}+1\cdot 2^{24}+\\ 0\cdot 2^{23}+1\cdot 2^{22}+0\cdot 2^{21}+1\cdot 2^{20}+1\cdot 2^{19}+1\cdot 2^{18}+1\cdot 2^{17}+1\cdot 2^{16}+\\ 1\cdot 2^{15}+1\cdot 2^{14}+1\cdot 2^{13}+1\cdot 2^{12}+1\cdot 2^{11}+0\cdot 2^{10}+1\cdot 2^{9}+0\cdot 2^{8}+\\ 1\cdot 2^{7}+1\cdot 2^{6}+1\cdot 2^{5}+1\cdot 2^{4}+1\cdot 2^{3}+0\cdot 2^{2}+1\cdot 2^{1}+0\cdot 2^{0}.\end{gathered} \tag{3.159}$$

In (3.159), as in (3.158), only the nonzero exponential terms contribute to the production of the integer since they have coefficient 1. The sum (3.159), omitting the zero terms, appears also in (3.150).

Also the binary integer in the third line of (3.157), which is the line $l\circ m$, analytically is

$$\begin{gathered}0\cdot 2^{31}+1\cdot 2^{30}+0\cdot 2^{29}+1\cdot 2^{28}+1\cdot 2^{27}+1\cdot 2^{26}+1\cdot 2^{25}+0\cdot 2^{24}+\\ 0\cdot 2^{23}+1\cdot 2^{22}+0\cdot 2^{21}+1\cdot 2^{20}+1\cdot 2^{19}+1\cdot 2^{18}+1\cdot 2^{17}+0\cdot 2^{16}+\\ 0\cdot 2^{15}+1\cdot 2^{14}+1\cdot 2^{13}+1\cdot 2^{12}+1\cdot 2^{11}+0\cdot 2^{10}+1\cdot 2^{9}+0\cdot 2^{8}+\\ 0\cdot 2^{7}+1\cdot 2^{6}+1\cdot 2^{5}+1\cdot 2^{4}+1\cdot 2^{3}+0\cdot 2^{2}+1\cdot 2^{1}+0\cdot 2^{0}.\end{gathered} \tag{3.160}$$

Similarly in (3.160) only the nonzero exponential terms are of interest.

10 Let us consider the procedure of the superposition of the lines as it appears, first in (3.157), and second in (3.158)–(3.160). In (3.157) we have a site-by-site superposition of the lines l and m to produce line $l\circ m$. There, when a site 1 of line l is superposed with a site 1 of line m then, and only then, the resulting site of line $l\circ m$ is 1. In any other case the resulting site of line $l\circ m$ is 0. We can see directly that the equivalent to the above procedure for (3.158)–(3.160) has as follows. The lines in (3.158)–(3.160) are represented as sums of the exponential terms $2^0, 2^1, 2^2, \cdots, 2^{31}$ each with coefficient 0 or 1. What we do for the superposition is just *to take the nonzero exponential terms (i.e., those with coefficient 1) which are common in (3.158) and (3.159) and to write them in (3.160) also as nonzero exponential terms (i.e., with coefficient 1). All the other* *exponential terms in (3.160) are zero terms (i.e., with coefficient 0). In this manner line (3.160) results.* Thus the common nonzero exponential terms in (3.158) and (3.159) are the terms

$$\begin{gathered} 2^{30}, 2^{28}, 2^{27}, 2^{26}, 2^{25}, 2^{22}, 2^{20}, 2^{19}, 2^{18}, 2^{17}, \\ 2^{14}, 2^{13}, 2^{12}, 2^{11}, 2^{9}, 2^{6}, 2^{5}, 2^{4}, 2^{3}, 2^{1} \end{gathered} \tag{3.161}$$

which are the nonzero exponential terms in (3.160). The other exponential terms in (3.160) are zero terms.

The numerical values $nz(l), nz(m)$, and $nz(l \circ m)$ are as follows. $nz(l)$ is equal to the total number of digits 1 in the first line of (3.157) or, equivalently, to the total number of nonzero exponential terms in (3.158), hence $nz(l) = 24$. $nz(m)$ is equal to the total number of digits 1 in the second line of (3.157) or, equivalently, to the total number of nonzero exponential terms in (3.159), hence $nz(m) = 24$. Finally, $nz(l \circ m)$ is equal to the total number of digits 1 in the third line of (3.157) or, equivalently, to the total number of nonzero exponential terms in (3.160) which is equal *to the total number of the nonzero exponential terms that are common in (3.158) and (3.159).* Hence $nz(l \circ m) = 20$.

Now Comment 3 has been adequately clarified through the above example.

11 In the just considered example the lines l and m, that is, the lines $2^2|f(1|1)|$ and $|f(1|2)|$ are totally elementary in the sense that they have only 32 sites with 0 or 1 (digits 0 or 1). Of course for the general lines $|f(i|j)|$, their multiples, and their superpositions it is unpractical to consider them analytically as sequences of digits 0 and 1 and, taking them as binary integers, to deduce from them the corresponding sums of the exponential terms of two, and so on. Instead we may use the general formulae for $|f(i|j)|$ and its multiples that have been developed through Chapters 2 and 3, see Tables 2.13–2.16 and Tables 3.5–3.7. There, lines $|f(i|j)|$ and their multiples are represented directly as functions of exponentials of two. From these functions it is much more easier, compared to the elementary representation of the lines $|f(i|j)|$ and their multiples (i.e., as sequences of digits 0 and 1) that has been used in the above example (the example clarifying Comment 3), to obtain lines $|f(i|j)|$ and their multiples as sums of exponential terms of two. Then we can use these sums of exponential terms to superpose the lines $|f(i|j)|$

and their multiples according to Comment 3. After all, what we were seeking during Chapter 2 and the first sections of Chapter 3 was a compact representation, through general formulae, of the lines $|f(i|j)|$ and their multiples. A representation permitting superpositions and the relevant computations through effective algorithms. The representation for $|f(i|j)|$ and its multiples, accomplished in Tables 2.13–2.16 and Tables 3.5–3.7, we expect that will facilitate the superpositions and all the computations concerning the lines.

12 In (3.90) we can see examples of lines $2^n|f(i|j)|$ expressed as functions of exponentials of two according to Tables 2.13–2.16 and to Tables 3.5–3.7. Similarly the lines $2^2|f(1|1)|$ and $|f(1|2)|$ of the example that has clarified Comment 3 are presented in (3.97) and (3.98) directly as functions of exponentials of two. From there we can derive directly the sums (3.158) and (3.159) respectively. Using the expressions in (3.97) and (3.98) we can easily find the superposition of $2^2|f(1|1)|$ and $|f(1|2)|$ according to Comment 3: analyse the functions of exponentials of two, in (3.97) and (3.98), into sums of exponential terms of two and then take the common such terms existing in the two sums, i.e., the sum obtained from (3.97) and the sum obtained from (3.98); the sum of these common terms represents, as we have seen discussing on Comment 3, the superposition of $2^2|f(1|1)|$ and $|f(1|2)|$ whereas the total number of these terms is the numerical value $nz[\,(2^2|f(1|1)|) \circ (|f(1|2)|)\,]$.

In Section 3.7, starting from (3.97) and (3.98), we derive the superposition of $2^2|f(1|1)|$ and $|f(1|2)|$ through manipulations performed during the whole section. Finally, at the end of the section just after (3.122), we have come to the (evident now) conclusion that the superposition of $2^2|f(1|1)|$ and $|f(1|2)|$ is determined from the intersection $(A \otimes B) \cap (C \otimes D)$ since $A \otimes B$ contains the exponential terms existing in $2^2|f(1|1)|$ and $C \otimes D$ contains the exponential terms existing in $|f(1|2)|$. Specifically we know now that the superposition is represented exactly by the sum of the exponential terms that are common in $2^2|f(1|1)|$ and in $|f(1|2)|$, i.e., by the sum of the exponential terms existing in the intersection $(A \otimes B) \cap (C \otimes D)$. This intersection appears, in proper notation, in (3.122) where we see which analytically are the common exponential terms and that their total number is twenty. Thus we see that $nz[\,(2^2|f(1|1)|) \circ (|f(1|2)|)\,] = 20$.

Now the discussion, announced at the last phrase of Section 3.7, is complete. Many things exposed in the present Section 3.8 could be deduced from the previous sections and chapters. But we considered good to include them in the three comments, as well as in the related discussion, and give them analytical descriptions. This was so because in the following work (computations, algorithms, manipulations, etc.) these things will be often used and, in fact, the whole work will be based on them. Thus it was necessary to clarify them in detail and entirely so that in the future, when we are engaged in complicated precedures, not be asking ourselves what exactly we are doing!

3.9 Continuation of Section 3.7.

Now, after the clarification of some important things in Section 3.8, we are ready to continue from the point we stopped at the end of Section 3.7.

1 We write again (3.117).

$$\begin{aligned} &A = \{b\,b\,b\,a\,a,\; b\,b\,b\,a\,b,\; b\,b\,b\,b\,a,\; b\,b\,b\,b\,b\}, \\ &B = \{21, 20, 10, 2, 1, 0\}. \end{aligned} \tag{3.162}$$

Being familiar with the notation and knowing well what is the meaning of A,B, and $A \otimes B$ we may write

$$A \otimes B = \{0A, 1A, 2A, 01A, 02A, 12A\} =$$

$$\begin{array}{lllll} \{a\,b\,b\,a\,a, & a\,b\,b\,a\,b, & a\,b\,b\,b\,a, & a\,b\,b\,b\,b, & 0A \\ b\,a\,b\,a\,a, & b\,a\,b\,a\,b, & b\,a\,b\,b\,a, & b\,a\,b\,b\,b, & 1A \\ b\,b\,a\,a\,a, & b\,b\,a\,a\,b, & b\,b\,a\,b\,a, & b\,b\,a\,b\,b, & 2A \\ a\,a\,b\,a\,a, & a\,a\,b\,a\,b, & a\,a\,b\,b\,a, & a\,a\,b\,b\,b, & 01A \\ a\,b\,a\,a\,a, & a\,b\,a\,a\,b, & a\,b\,a\,b\,a, & a\,b\,a\,b\,b, & 02A \\ b\,a\,a\,a\,a, & b\,a\,a\,a\,b, & b\,a\,a\,b\,a, & b\,a\,a\,b\,b\,\} & 12A \end{array} \tag{3.163}$$

In the set in the second part of (3.163) by e.g. $12A$ we mean the set of the products of the string (exponential term) 12 with all strings (exponential terms) in the set A, see (3.162) and also (3.117)–(3.121). It is clear that the second part of (3.163) is written instead of $\{A21, A20, A10, A2, A1, A0\}$ which is equivalent. In the third part of (3.163) we have a set in block-form. The column with $0A, 1A, 2A, \cdots$ at the right side of the block is not part of the block-set but it is just an elucidation: the first row of the block-set corresponds to $0A$, the second row corresponds to $1A$, and so on.

Similarly for C and D we write again (3.118).

$$\begin{aligned} C &= \{b\,a\,b\,a\,b,\ b\,a\,b\,b\,b,\ b\,b\,b\,a\,b,\ b\,b\,b\,b\,b\}, \\ D &= \{42, 40, 20, 4, 2, 0\}. \end{aligned} \tag{3.164}$$

Then it is

$$C \otimes D = \{0C, 2C, 4C, 02C, 04C, 24C\} =$$

$$\begin{array}{llllc} \{a\,a\,b\,a\,b, & a\,a\,b\,b\,b, & a\,b\,b\,a\,b, & a\,b\,b\,b\,b, & 0C \\ \ b\,a\,a\,a\,b, & b\,a\,a\,b\,b, & b\,b\,a\,a\,b, & b\,b\,a\,b\,b, & 2C \\ \ b\,a\,b\,a\,a, & b\,a\,b\,b\,a, & b\,b\,b\,a\,a, & b\,b\,b\,b\,a, & 4C \\ \ a\,a\,a\,a\,b, & a\,a\,a\,b\,b, & a\,b\,a\,a\,b, & a\,b\,a\,b\,b, & 02C \\ \ a\,a\,b\,a\,a, & a\,a\,b\,b\,a, & a\,b\,b\,a\,a, & a\,b\,b\,b\,a, & 04C \\ \ b\,a\,a\,a\,a, & b\,a\,a\,b\,a, & b\,b\,a\,a\,a, & b\,b\,a\,b\,a\,\} & 24C \end{array} \tag{3.165}$$

2 For set $A \otimes B$ a basic fact is that it contains strings of length 5 (i.e., with five letters) and the sites are 0,1,2,3,4. From these the special ones are sites 0,1,2. Similarly for set $C \otimes D$ a basic fact is that it contains strings of length 5 and the sites are 0,1,2,3,4. From these the special ones are sites 0,2,4.

The strings of $A \otimes B$, appearing analytically in the third part of (3.163), may be separated into six classes according to the letters existing in the special sites 0,1,2. We present these classes in (3.166).

$$\begin{array}{lccc} & 0, & 1, & 2 \\ \text{class 1 :} & a & b & b \\ \text{class 2 :} & b & a & b \\ \text{class 3 :} & b & b & a \\ \text{class 4 :} & a & a & b \\ \text{class 5 :} & a & b & a \\ \text{class 6 :} & b & a & a \end{array} \tag{3.166}$$

It is clear that: class 1 contains the strings existing in the first row of the set in the third part of (3.163) [the row that corresponds to $0A$]; class 2 contains the strings existing in the second row of the set in the third part of (3.163) [the row that corresponds to $1A$];

and so on.

We work similarly for $C \otimes D$. The strings of $C \otimes D$, appearing analytically in the third part of (3.165), may be separated into six classes according to the letters existing in the special sites 0,2,4. These classes are presented in (3.167).

$$\begin{array}{lccc} & 0, & 2, & 4 \\ \text{class 1 :} & a & b & b \\ \text{class 2 :} & b & a & b \\ \text{class 3 :} & b & b & a \\ \text{class 4 :} & a & a & b \\ \text{class 5 :} & a & b & a \\ \text{class 6 :} & b & a & a \end{array} \tag{3.167}$$

Similarly here it is clear that: class 1 contains the strings existing in the first row of the set in the third part of (3.165) [the row that corresponds to $0C$]; class 2 contains the strings existing in the second row of the set in the third part of (3.165) [the row that corresponds to $2C$]; and so on.

In (3.166) and (3.167) we present the possible combinations for the special sites 0,1,2 and 0,2,4 that correspond respectively to the strings of $A \otimes B$ and $C \otimes D$. The other sites of the strings contain all possible combinations of letters a and b.

3 Let us see now how the classes in (3.166) and (3.167) can help in finding the common strings of $A \otimes B$ and $C \otimes D$. These common strings, i.e. the intersection $(A \otimes B) \cap (C \otimes D)$, are important, as we have seen in Sec. 3.8, for determining the superposition of the lines.

We denote each of the six classes in (3.166) as $1_{AB}, 2_{AB}, \cdots, 6_{AB}$ and each of the six classes in (3.167) as $1_{CD}, 2_{CD}, \cdots, 6_{CD}$. Wishing to find the common strings of $A \otimes B$ and $C \otimes D$ we consider separately their six classes, we find the common strings among these classes, and finally we find the common strings of $A \otimes B$ and $C \otimes D$ by summing up the common strings of the separate classes. Let us see it analytically.

From (3.166) we have that the strings of class 1_{AB} in sites 0,1,2 contain letters a, b, b respectively. Thus strings of this class may exist, in set $C \otimes D$, only in class 1_{CD} [which in sites 0,2,4 contains letters a, b, b respectively, see (3.167)] and in class 5_{CD} [which in sites 0,2,4 contains letters a, b, a respectively, see (3.167)]. This is so because the strings of class 1_{AB} are compatible with the strings of class 1_{CD} and class 5_{CD} (we say that class 1_{AB} is compatible with classes 1_{CD} and 5_{CD}) in the sense that sites 0,1,2 of class 1_{AB} being a, b, b are compatible with sites 0,2,4 being a, b, b in class 1_{CD} and a, b, a in class 5_{CD}. The compatibility has to do with the fact that sites 0,1,2 and 0,2,4 have common the sites 0,2 which are a, b for all classes $1_{AB}, 1_{CD}, 5_{CD}$. In contrast, as an example of incompatible classes we have classes 1_{AB} and 2_{CD}. Indeed the strings of class 1_{AB} having in their sites 0,1,2 letters a, b, b are incompatible (cannot be common) with the strings of class 2_{CD} having in their sites 0,2,4 letters b, a, b. That is so because in the common sites 0,2 in class 1_{AB} we have a, b whereas in class 2_{CD} we have b, a and this is incompatibility since, in a string, site 0 cannot be a and b at the same time and similarly site 2 cannot be b and a at the same time. Thus, according to the above description, class 1_{AB} has with class 1_{CD} common the strings which in their sites 0,1,2,3,4 (and these are all their sites) have letters $a, b, b, *, b$ with symbol $*$ in site 3 being a or b; so the total number of common strings is two and these analytically are *abbab* and *abbbb*. Similarly class 1_{AB} has with class 5_{CD} common the strings which in their sites 0,1,2,3,4 have $a, b, b, *, a$ with $*$ being a or b; so we have totally two common strings namely *abbaa* and *abbba*.

4 All the above compactly may be written as follows:

<u>Class 1_{AB}</u>:	$\begin{matrix} 0 & 1 & 2 \\ a & b & b \end{matrix}$	compatible (has common strings) with		
class 1_{CD}:	$\begin{matrix} 0 & 2 & 4 \\ a & b & b \end{matrix}$	with 2 common strings	$\begin{matrix} 0 & 1 & 2 & 3 & 4 \\ a & b & b & * & b \end{matrix}$	
class 5_{CD}:	$\begin{matrix} 0 & 2 & 4 \\ a & b & a \end{matrix}$	with 2 common strings	$\begin{matrix} 0 & 1 & 2 & 3 & 4 \\ a & b & b & * & a \end{matrix}$	

What we did for class 1_{AB} we can do for the other classes $2_{AB}, \cdots, 6_{AB}$ as well. So in exactly similar manner we obtain:

<u>Class 2_{AB}</u>: $\begin{matrix} 0 & 1 & 2 \\ b & a & b \end{matrix}$ compatible (has common strings) with

class 3_{CD}: $\begin{array}{ccc}0&2&4\\ b&b&a\end{array}$ with 2 common strings $\begin{array}{ccccc}0&1&2&3&4\\ b&a&b&*&a\end{array}$

Class 3_{AB}: $\begin{array}{ccc}0&1&2\\ b&b&a\end{array}$ compatible (has common strings) with

class 2_{CD}: $\begin{array}{ccc}0&2&4\\ b&a&b\end{array}$ with 2 common strings $\begin{array}{ccccc}0&1&2&3&4\\ b&b&a&*&b\end{array}$

class 6_{CD}: $\begin{array}{ccc}0&2&4\\ b&a&a\end{array}$ with 2 common strings $\begin{array}{ccccc}0&1&2&3&4\\ b&b&a&*&a\end{array}$

Class 4_{AB}: $\begin{array}{ccc}0&1&2\\ a&a&b\end{array}$ compatible (has common strings) with

class 1_{CD}: $\begin{array}{ccc}0&2&4\\ a&b&b\end{array}$ with 2 common strings $\begin{array}{ccccc}0&1&2&3&4\\ a&a&b&*&b\end{array}$

class 5_{CD}: $\begin{array}{ccc}0&2&4\\ a&b&a\end{array}$ with 2 common strings $\begin{array}{ccccc}0&1&2&3&4\\ a&a&b&*&a\end{array}$

Class 5_{AB}: $\begin{array}{ccc}0&1&2\\ a&b&a\end{array}$ compatible (has common strings) with

class 4_{CD}: $\begin{array}{ccc}0&2&4\\ a&a&b\end{array}$ with 2 common strings $\begin{array}{ccccc}0&1&2&3&4\\ a&b&a&*&b\end{array}$

Class 6_{AB}: $\begin{array}{ccc}0&1&2\\ b&a&a\end{array}$ compatible (has common strings) with

class 2_{CD}: $\begin{array}{ccc}0&2&4\\ b&a&b\end{array}$ with 2 common strings $\begin{array}{ccccc}0&1&2&3&4\\ b&a&a&*&b\end{array}$

class 6_{CD}: $\begin{array}{ccc}0&2&4\\ b&a&a\end{array}$ with 2 common strings $\begin{array}{ccccc}0&1&2&3&4\\ b&a&a&*&a\end{array}$

5 Thus we see analytically above that: class 1_{AB} has two common strings with class 1_{CD}, and two common strings with class 5_{CD}; class 2_{AB} has two common strings with class 3_{CD}; and so on. Consequently: class 1_{AB} has totally four (2+2) common strings with set

$C \otimes D$; class 2_{AB} has totally two common strings with set $C \otimes D$; and so on. Finally we deduce that: set $A \otimes B$ (which is the union of the six classes $1_{AB}, 2_{AB}, \cdots, 6_{AB}$) has totally 20 (4+2+4+4+2+4) common strings with set $C \otimes D$ a fact already well known, see for example (3.120)–(3.122).

6 The special sites 0,1,2 for $A \otimes B$ [which means for $2^2|f(1|1)|$] and 0,2,4 for $C \otimes D$ [which means for $|f(1|2)|$] are specific cases of the three basic sites

$$i-1, i+j-1, i+2j-1 \tag{3.168}$$

[which are for $2^n|f(i|j)|$] for $(i,j) = (1,1)$ and $(i,j) = (1,2)$ respectively, see the last phrase of Sec. 3.7,Subsec. 2 just after (3.109). What we have done for the example with the superposition of $2^2|f(1|1)|$ and $|f(1|2)|$ may be generalized for the superposition of $2^n|f(i|j)|$ and $2^{n'}|f(i'|j')|$. Taking into account the detailed analysis performed for $2^n|f(i|j)|$ in the sections following Table 3.8 and (3.42) we work as below.

For each triple (i,j,n), i.e. for each line $2^n|f(i|j)|$, we consider strings of length

$$i+2j+n, \tag{3.169}$$

which means that they have sites

$$0, 1, 2, \cdots, i+2j+n-1. \tag{3.170}$$

Superposing lines $2^n|f(i|j)|$ and $2^{n'}|f(i'|j')|$, since the superposed strings must have same length, it is always

$$i'+2j'+n'-1 = i+2j+n-1. \tag{3.171}$$

In accordance with the examples of $2^2|f(1|1)|$ and $|f(1|2)|$, for which we defined sets $A, B, A \otimes B$ and $C, D, C \otimes D$ respectively, we define similar sets for the general cases of $2^n|f(i|j)|$ and $2^{n'}|f(i'|j')|$. Specifically, for $2^n|f(i|j)|$, in accordance with the definition of A in (3.115) and (3.117), we have

$$A = \begin{array}{l}\{\text{all strings of } i+2j+n \text{ letters (sites } 0,1,2,\cdots, \\ i+2j+n-1) \text{ for which: sites } i-1, i+j-1, \\ i+2j-1 \text{ are } b\text{; the other sites are all possible} \\ \text{combinations of letters } a \text{ and } b\}\end{array} \tag{3.172}$$

Also, in accordance with the definition of B in (3.115),(3.117), and (3.162), we have

$$\begin{aligned} B = \ & \{i-1, i+j-1, i+2j-1, (i-1)(i+j-1), \\ & (i-1)(i+2j-1), (i+j-1)(i+2j-1)\} \end{aligned} \qquad (3.173)$$

Similarly, in accordance with (3.163), we have

$$\begin{aligned} A \otimes B = \ & \{(i-1)A, (i+j-1)A, (i+2j-1)A, \\ & (i-1)(i+j-1)A, (i-1)(i+2j-1)A, \\ & (i+j-1)(i+2j-1)A\} = \end{aligned}$$

$$\begin{array}{lr} \{expression(a,b,b), & (i-1)A \\ \ expression(b,a,b), & (i+j-1)A \\ \ expression(b,b,a), & (i+2j-1)A \\ \ expression(a,a,b), & (i-1)(i+j-1)A \\ \ expression(a,b,a), & (i-1)(i+2j-1)A \\ \ expression(b,a,a)\ \} & (i+j-1)(i+2j-1)A \end{array} \qquad (3.174)$$

where e.g. $expression(a,b,b)$ is a brief manner to denote the expression that is presented below:

$$\begin{aligned} expression(a,b,b) \equiv \ & (\text{all strings with length } i+2j+n \text{ and} \\ & \text{sites } 0,1,2,\cdots,i+2j+n-1 \text{ from} \\ & \text{which: sites } i-1, i+j-1, i+2j-1 \\ & \text{have letters } a,b,b\text{; the other sites} \\ & \text{contain all possible combinations} \\ & \text{of letters } a \text{ and } b) \end{aligned} \qquad (3.175)$$

Exactly as in (3.163), in the third part of (3.174) there is a correspondence between the rows of the set, i.e. $expression(a,b,b), expression$ $(b,a,b), etc.$, and $(i-1)A, (i+j-1)A, etc.$ Besides, (3.163) [which is for $2^2|f(1|1)|$] results directly from (3.174) and (3.175) [which are for $2^n|f(i|j)|$] if we put $(i,j,n) = (1,1,2)$. Also, putting $(i,j,n) = (1,2,0)$ into (3.174) and (3.175) [and C and D instead of A and B] we obtain directly (3.165) which is for $|f(1|2)|$.

For $2^{n'}|f(i'|j')|$ we can do similar things as for $2^n|f(i|j)|$. In this case we define sets C and D which equally may be called A' and B' respectively.

7 Now we can obtain the superposition of lines $2^n|f(i|j)|$ and $2^{n'}|f(i'|j')|$ in the same manner, i.e. by a direct generalization, as we obtained the superposition of lines $2^2|f(1|1)|$ and $|f(1|2)|$. For simplicity the three special sites are denoted k, l, m and k', l', m' which means that we have by definition

$$(i-1, i+j-1, i+2j-1) = (k, l, m), \tag{3.176a}$$

and

$$(i'-1, i'+j'-1, i'+2j'-1) = (k', l', m'). \tag{3.176b}$$

Working as for $2^2|f(1|1)|$ and $|f(1|2)|$ we obtain, for $2^n|f(i|j)|$ and $2^{n'}|f(i'|j')|$, sets $A \otimes B$ and $C \otimes D$ respectively. The strings in these two sets are separated into six classes, in exactly same way as we have seen in (3.166) and (3.167), but instead of the special sites 0,1,2 and 0,2,4 we have now the generalized special sites k, l, m and k', l', m' respectively. The classes are determined from the kind of letters existing in the triples of these special sites.

So analytically, in accordance with (3.166), we have now that the strings in the set $A \otimes B$, which is obtained for $2^n|f(i|j)|$, are separated into six classes as follows:

$$\begin{array}{lccc} & k, & l, & m \\ \text{class } 1: & a & b & b \\ \text{class } 2: & b & a & b \\ \text{class } 3: & b & b & a \\ \text{class } 4: & a & a & b \\ \text{class } 5: & a & b & a \\ \text{class } 6: & b & a & a \end{array} \tag{3.177}$$

Similarly the strings in the set $C \otimes D$, which is for $2^{n'}|f(i'|j')|$, in accordance with (3.167) are separated into six classes as follows:

$$\begin{array}{lccc} & k', & l', & m' \\ \text{class } 1: & a & b & b \\ \text{class } 2: & b & a & b \\ \text{class } 3: & b & b & a \\ \text{class } 4: & a & a & b \\ \text{class } 5: & a & b & a \\ \text{class } 6: & b & a & a \end{array} \tag{3.178}$$

Putting $(i, j) = (1, 1)$ into (3.176a) we obtain $(k, l, m) = (0, 1, 2)$ and (3.177) becomes (3.166). Also putting $(i', j') = (1, 2)$ into

(3.176b) we obtain $(k', l', m') = (0, 2, 4)$ and (3.178) becomes (3.167).

If we wish to find the superposition of the lines $2^n|f(i|j)|$ and $2^{n'}|f(i'|j')|$ we have to find, as we know, the common strings of sets $A \otimes B$ and $C \otimes D$ which means to find the intersection $(A \otimes B) \cap (C \otimes D)$. For this purpose we may use the separation of sets $A \otimes B$ and $C \otimes D$ into the classes appearing in (3.177) and (3.178). We work as we have seen in Subsections 3–5 of the present Section 3.9 regarding the examples of (3.166) and (3.167). But here all (strings, special sites, length of strings, etc.) are taken in their generalized version.

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