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SUBJECT C.

Quantum Groups, Conformal
field theory and noncommu-
tative geometry.

by

Vassilis Konstantopoulos.

First article

Correlation functions for a WZW theory and representation of quasitriangular Hopf algebras - Part one

Abstract

For a finite dimensional complex simple Lie algebra \mathfrak{g} we consider the associated standard Hopf QVE algebra, we examine the equivalence according to the theory of Kohno and Drinfeld of the two representations of the braid group P_{KZ} and P_{AYBE} . We consider the KZ equation as derived in the WZW conformal field theory and in the affine Lie algebras theory.

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1. Braid group, braided vector space

definition 1 (Artin's braid group)

Consider an integer $n \geq 3$. The braid group B_n on n strands, is the group with $n-1$ generators $\sigma_1, \dots, \sigma_{n-1}$ and relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| > 1 \quad (a) \quad (7-1)$$

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \text{ for } |i-j| = 1 \quad (b)$$

we can write

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } 1 \leq i \leq n-1 \quad (b)'$$

we define for $n=2$ the braid group B_2 as the free group with one generator and we let

$$B_0 = B_1 = \{1\}$$

be the trivial group. (1-2)

definition 2

Let K be a field.

A braided vector space, is a vector space V together with an invertible K -linear map

$$C: V \otimes V \rightarrow V \otimes V$$

which obeys the equation:

$$\begin{aligned}
 (C \otimes id_V) \circ (id_V \otimes C) \circ (C \otimes id_V) &= \\
 = (id_V \otimes C) \circ (C \otimes id_V) \circ (id_V \otimes C) & \quad (1-3)
 \end{aligned}$$

in $End(V \otimes V \otimes V)$
 id_V denotes the identity map $V \rightarrow V$
 $End(V \otimes V \otimes V)$ denotes the endomorphisms $V \otimes V \otimes V \rightarrow V \otimes V \otimes V$.

Corollary 1.

Let $(U_i)_{i \in I}$ be a K -basis of V .
 This allows us to describe $C \in End(V \otimes V)$
 by a family $(C_{ij}^{kl})_{i,j,k,l \in I}$

of scalars: k, l
 $C(U_i \otimes U_j) = \sum_{k,l} C_{ij}^{kl} U_k \otimes U_l \quad (1-4)$

Corollary 2.

If C is invertible, then C describes
 an isomorphism of vector space if and only
 if, the following equation holds;

$$\sum_{p,q} C_{ij}^{pq} C_{qk}^{yn} C_{py}^{lm} = \sum_{y,q,r} C_{jk}^{qr} C_{iq}^{ly} C_{yr}^{mn} \quad (1-5)$$

$\forall l, m, n, i, j, k \in I$
 I is the set of indices.

Corollary 3.

Let (V, c) be a braided vector space
for $1 \leq i \leq n-1$, define an auto-
morphism of $V^{\otimes n}$, by

$$C_i \stackrel{\Delta}{=} \left[\begin{array}{ll} C \otimes \text{id}_{V^{\otimes (n-2)}} & \text{for } i=1 \\ \text{id}_{V^{\otimes (i-1)}} \otimes C \otimes \text{id}_{V^{\otimes (n-i-1)}} & \text{for } 1 < i < n-1 \\ \text{id}_{V^{\otimes (n-2)}} \otimes C & \text{for } i=n-1 \end{array} \right] \quad (1-5)$$

For any $n > 0$ we have a unique
homomorphism of groups

$$\rho_n^C : B_n \rightarrow \text{Aut}(V^{\otimes n}) \quad (1-6)$$

$$b_i \mapsto C_i \quad \text{for } i=1, 2, \dots, n-1 \quad (1-7)$$

Note

we write $V^{\otimes n}$ for

$$\underbrace{V \otimes V \otimes \dots \otimes V}_{n \text{ - times}}$$

$\text{id}_{V^{\otimes n}}$ is the identity map for the
space $V^{\otimes n}$

B_n is the braid group on n -strands.

Proof:

In fact the relations

$$C_i C_j = C_j C_i \text{ for } |i-j| \geq 2$$

holds, since the linear maps C_i, C_j act on different copies of the tensor product.

The relation

$$C_i C_{i+1} C_i = C_{i+1} C_i C_{i+1}$$

is part of the axioms of a braid vector spaces.

2. Quantum groups and representation of braid groups.

Definition 3 Hopf algebra.

A Hopf algebra A , is a vector space endowed with five operations

$$m: A \otimes A \rightarrow A \quad \text{multiplication}$$

$$u: F \rightarrow A \quad \text{unit map}$$

$$\Delta: A \rightarrow A \otimes A \quad \text{co-multiplication}$$

$$\varepsilon: A \rightarrow F \quad \text{co-unit map}$$

$$S: A \rightarrow A \quad \text{antipode}$$

which possess the following properties

1. $m \circ (1_d \otimes m) = m \circ (m \otimes 1_d)$
associativity

2. $m \circ (1_d \otimes u) = m \circ (u \otimes 1_d)$
existence of unit.

3. $(1_d \otimes \Delta) \circ \Delta = (\Delta \otimes 1_d) \circ \Delta$
co-associativity

4. $(\varepsilon \otimes 1_d) \circ \Delta = (1_d \otimes \varepsilon) \circ \Delta = 1_d$
existence of counity

5. connecting axiom

5. $\Delta \circ m = (m \otimes m) \circ (\Delta \otimes \Delta)$
6. Existence of antipode

$$S: A \rightarrow A$$

$$m \circ (1_d \otimes S) \circ \Delta = \eta \circ \epsilon = m \circ (S \otimes 1_d) \circ \Delta$$

Note.

We consider A as a vector space over a zero characteristic field F .

1_d is the identity $A \rightarrow A$.

Remark 1

For the multiplication m we have

$$m: A \otimes A \rightarrow A, \forall a, b \in A \quad m(a \otimes b) \stackrel{\Delta}{=} a \cdot b \in A$$

For the associativity we have

$$x \cdot m(y \cdot z) = (x \cdot y) \cdot z, \forall x, y, z \in A. \quad (1-8)$$

$\stackrel{\Delta}{=}$ means: we define as...

Remark 2

For the tensor product of two vector spaces A, B we have

for $f: A \rightarrow A$, or linear map,
and $g: B \rightarrow B$, " " "

$$f \otimes g: A \otimes B \rightarrow A \otimes B$$

$$\forall a, b, a \in A, b \in B \quad (f \otimes g)(a \otimes b) =$$

$$= f(a) \otimes f(b) \quad (1-9)$$

For the algebra A we have

$$\begin{aligned} \forall x, y, z \in A \quad (Id \otimes m)(x \otimes y \otimes z) &= \\ &= x \otimes ymz \\ m \circ (Id \otimes m)(x \otimes y \otimes z) &= m(x \otimes ymz) = \\ &= x m(y m z) \quad \text{e.t.c.} \end{aligned}$$

Remark 3.

There is a nice notation for Δ ,
we write

$$\Delta(c) = \sum_i C_{i(1)} \otimes C_{i(2)} \quad (1-10)$$

$$\Delta: A \rightarrow A \otimes A$$

The right hand side is a formal sum denoting an element of $A \otimes A$.

we simply write

$$\Delta(c) = \sum C_{(1)} \otimes C_{(2)} \quad (1-11)$$

coassociativity then means that if we share out again, it does not matter which piece of $\Delta(c)$ we share out.

Thus we write

$$\begin{aligned} C_{(1)} \otimes C_{(2)(1)} \otimes C_{(2)(2)} &= C_{(1)(1)} \otimes C_{(1)(2)} \otimes C_{(2)} = \\ &= C_{(1)} \otimes C_{(2)} \otimes C_{(3)} \quad (1-12) \end{aligned}$$

we can write also, for the co-multiplication

$$\Delta: A \rightarrow A \otimes A$$

$$\forall x \in A \quad x \mapsto \Delta(x) = \sum_e x_1^{(e)} \otimes x_2^{(e)} \quad (1-13)$$

we have

$$\Delta(xmy) = \sum_{e,m} (x_1^{(e)} m y_1^{(m)}) \otimes (x_2^{(e)} m y_2^{(m)}) \quad (1-14)$$

The comultiplication and co-unit are homomorphisms of A i.e. they preserve the algebra multiplication

$$\Delta(xmy) = \Delta(x) m \Delta(y), \quad \forall x, y \in A$$

For the antipode we have

$$S(xmy) = S(y) m S(x), \quad \forall x, y \in A \quad (1-15)$$

that is the antipode S is an anti-homomorphism.

definition 4.

$$\pi: A \otimes A \rightarrow A \otimes A$$

$$\pi(x \otimes y) \stackrel{\Delta}{=} y \otimes x, \quad \forall x, y \in A \quad (1-16)$$

we say that π is a permutation

corollary 4:

The map

$$\Delta' = \pi \circ \Delta \quad (1-17)$$

is also a co-associative, co-multiplication

- Contin.

Remark 4.

For a commutative algebra we have

$$m \circ \pi = m \tag{1-18}$$

a Hopf algebra A , analogously, is called co-commutative if

$$\pi \circ \Delta = \Delta \tag{1-19}$$

or in other words if Δ' coincides with Δ .

definition 5.

Let A be a Hopf algebra, an element $R \in A \otimes A$ has an inverse if

$$R^{-1} m R = e \otimes e = R m R^{-1} \tag{1-20}$$

e is the unit, we can extend the algebra multiplication m , we use the same symbol m for the multiplication in $T(A)$,

$$T(A) = A \otimes \dots \otimes A, \text{ etc.}$$

definition 6.

Let A be a Hopf algebra, consider an element $R \in A \otimes A$, of the form

$$R = v_1 \otimes v_2, v_1, v_2 \in A \tag{1-21}$$

we define

$$R_{13} \in A \otimes A \otimes A, \quad R_{13} = r_1 \otimes e \otimes v_2$$

$$R_{12} \in A \otimes A \otimes A, \quad R_{12} = r_1 \otimes v_2 \otimes e \quad (1-22)$$

$$R_{23} \in A \otimes A \otimes A, \quad R_{23} = e \otimes r_1 \otimes v_2$$

e is the unit in A

definition 7.

In the general case, $R \in A \otimes A$
and $R \neq r_1 \otimes v_2$ we can write

$$R = \sum_e r_1^{(e)} \otimes v_2^{(e)}$$

and we can define similarly

$$R_{13} = \sum_e r_1^{(e)} \otimes e \otimes v_2^{(e)}, \quad R_{13} \in A \otimes A \otimes A$$

$$R_{12} = \sum_e r_1^{(e)} \otimes v_2^{(e)} \otimes e, \quad R_{12} \in A \otimes A \otimes A \quad (1-23)$$

$$R_{23} = \sum_e e \otimes r_1^{(e)} \otimes v_2^{(e)}, \quad R_{23} \in A \otimes A \otimes A.$$

definition 8.

A quasi-triangular Hopf algebra
is a pair (A, R) where A is a Hopf
algebra and there exists:

$R \in A \otimes A$, R invertible with the
properties

for $\Delta' = \pi_0 \Delta$

$$\Delta'(x) = R_m \Delta(x) m R^{-1}, \quad \forall x \in A \quad (1-24)$$

i.e. the co-multiplications Δ, Δ' are related by conjugacies

$$(id \otimes \Delta)(R) = R_{13} m R_{12} \quad (a) \quad (1-25)$$

$$(\Delta \otimes id)(R) = R_{13} m R_{23} \quad (b)$$

Note.

$$(S \otimes id)R = R^{-1} \quad (1-26)$$

definition 9.

A quasi-triangular Hopf algebra is called triangular if and only if

$$R_{12} m R_{21} = e \otimes e \quad (1-27)$$

or in other words if and only if

$$\pi(R) = R^{-1}.$$

Remark 5.

we have the quantum Yang-Baxter equation (Q.Y.B.E)

$$R_{12} m R_{13} m R_{23} = R_{23} m R_{13} m R_{12} \quad (1-28)$$

proposition 7

Suppose that (H, R) is a quasi-triangular Hopf algebra, we define the operators

$$C_i: V^{\otimes m} \rightarrow V^{\otimes m}$$

by setting

$$C_i(u_1 \otimes \dots \otimes u_m) =$$

$$= u_1 \otimes \dots \otimes u_{i-1} \otimes \sigma(u_i \otimes u_{i+1}) \otimes u_{i+2} \otimes \dots \otimes u_m,$$

$$\text{---} \otimes u_m, \quad (1-29)$$

where $\sigma \in \text{End}(V \otimes V)$ is the interchange of the two factors.

In this way we obtain a representation of a Braid group B_m on $V^{\otimes m}$

$$u_1, \dots, u_m \in V, \quad V^{\otimes m} = \underbrace{V \otimes \dots \otimes V}_{m \text{ times}}$$

proof.

Indeed it is easy to check that the QYBE is equivalent to the braid group relations

$$C_i C_j C_i = C_j C_i C_j \quad (1-30)$$

if $|i-j|=1$

and

$$C_i C_j = C_j C_i, \text{ if}$$

$$|i-j| > 1$$

This is obvious because if $|i-j| > 1$

the operators ρ , C_i and C_j act on disjoint pairs of factors in $V^{\otimes m}$.

definition 10

Consider the quasi-triangular Hopf algebra (V, R) and a braid group B_m on m strands and generators τ_i .

We write

$$\rho^{QTB E}(\tau_i) = C_i \quad (1-31)$$

as defined in (1-29).

$\rho^{QTB E}$ is a representation of B_m

Note.

V as a vector space, is a braided vector space.

We observe that given a quasi-triangular Hopf algebra (A, R) and a braid group on m -strands we construct the QTB E - representation: $\rho^{QTB E}$.

3. deformations of Hopf algebras

definition 11.

Let A be an associative algebra, a deformation of A is a family of bilinear associative maps

$$*_h : A \times A \rightarrow A$$

depending on a parameter h , such that

$$*_0 : A \times A \rightarrow A$$

is the given multiplication of A .

Note.

we consider only formal deformations, this means that we treat h as an indeterminate and expand $*_h$ in a formal power series

$$\alpha_1 *_h \alpha_2 = \sum_{n=0}^{\infty} h^n \pi_n(\alpha_1, \alpha_2) \quad (1-32)$$

for certain bilinear maps

$$\pi_n : A \times A \rightarrow A$$

The associativity of $*_h$ is equivalent to

$$\sum_{r+s=n} \pi_r(\pi_s(\alpha_1, \alpha_2), \alpha_3) = \quad (1-33)$$

$$= \sum_{r+j=n} \pi_r(\alpha_1, \pi_j(\alpha_2, \alpha_3))$$

for all $\alpha_1, \alpha_2, \alpha_3 \in A$, $n \geq 0$

definition 12

1. Consider an algebra A . We write $A[[h]]$ for the algebra of formal power series in h with coefficients in A .

that is

$$\alpha_h \in A[[h]]$$

(1-34)

$$\alpha_h = \alpha_0 + \alpha_1 h + \alpha_2 h^2 + \dots$$

for $\alpha_0, \alpha_1, \alpha_2, \dots \in A$

2. Let K be a field of characteristic zero.

$K[[h]]$ is the ring of formal power series in an indeterminate h over the field K .

definition 13

A topological Hopf algebra over $K[[h]]$ is a complete $K[[h]]$ -module A , equipped with $K[[h]]$ -linear maps $\mu, \Delta, i, \varepsilon, S$ satisfying the axioms for the definition of a Hopf algebra.

definition 14: deformation of a Hopf algebra

A deformation of a Hopf algebra
 $(A, \mu, \Delta, i, \varepsilon, S)$ over a field K is
 a topological Hopf algebra $(A_h, \mu_h, \Delta_h,$
 $i_h, \varepsilon_h, S_h)$
 over the ring $K[[h]]$ such that

1. A_h is isomorphic to $A[[h]]$
 as a $K[[h]]$ module

2. For $\alpha, \alpha' \in A$, there are K -module
 maps

$$\mu_h : A \otimes A \rightarrow A$$

$$\Delta_h : A \rightarrow A \otimes A$$

such that:

$$\begin{aligned} \mu_h(\alpha \otimes \alpha') &= \mu(\alpha \otimes \alpha') + \mu_1(\alpha \otimes \alpha')h + \\ &+ \mu_2(\alpha \otimes \alpha')h^2 + \dots \end{aligned} \quad (1-35)$$

$$\Delta_h(\alpha) = \Delta(\alpha) + \Delta_1(\alpha)h + \Delta_2(\alpha)h^2 + \dots$$

we write (1-36)

$$\mu_h \equiv \mu \pmod{h}, \quad \Delta_h \equiv \Delta \pmod{h}$$

definition 15:

Two deformations A_h, A'_h are

equivalent if there is an isomorphism

$$f_h : A_h \rightarrow A'_h$$

of Hopf algebras over $K[[h]]$

which is the identity (mod h)

$$f_h = I \pmod{h} \quad (1-37)$$

that mean

$$f_h(\alpha) = \alpha + hf_1(\alpha) + h^2 f_2(\alpha) + \dots \quad (1-38)$$

$$f_i : A \rightarrow A$$

are isomorphisms of Hopf algebras

Note.

$$f_h^{-1}(\alpha) = \alpha - hf_1(\alpha) + \dots \quad (1-39)$$

$$\alpha \in A$$

Corollary 4

The associativity condition for μ_h is

$$\begin{aligned} \mu_h(\mu_h(\alpha_1 \otimes \alpha_2) \otimes \alpha_3) &= \\ &= \mu_h(\alpha_1 \otimes \mu_h(\alpha_2 \otimes \alpha_3)) \end{aligned} \quad (1-40)$$

Similarly the coassociativity condition for Δ_h is

$$(\Delta_h \otimes \text{id}) \Delta_h(\alpha) = (\text{id} \otimes \Delta_h) \Delta_h(\alpha) \quad (1-41)$$

$$\alpha_1, \alpha_2, \alpha \in A$$

writing both sides of this equation as formal power series in \hbar , we get an infinite number of conditions on the components μ_i of μ_\hbar and Δ_i of Δ_\hbar .

The components of order n gives

$$\begin{aligned} \mu_n(a_1 a_2 \otimes a_3) + \mu_n(a_1 \otimes a_2) a_3 &= \\ = a_1 \mu_n(a_2 \otimes a_3) + \mu_n(a_1 \otimes a_2 a_3) & \quad (1-42). \end{aligned}$$

and

$$\begin{aligned} (\Delta \otimes \text{id}) \Delta_1(a) + (\Delta_1 \otimes \text{id}) \Delta(a) &= \\ = (\text{id} \otimes \Delta) \Delta_1(a) + (\text{id} \otimes \Delta_1) \Delta(a) & \quad (1-43). \end{aligned}$$

we write a_1, a_2 for $\mu(a_1 \otimes a_2)$

Corollary 5:

The conditions for a $K[[\hbar]]$ -module isomorphism

$$f_\hbar: A_\hbar \rightarrow A'_\hbar$$

to be an equivalence of deformations are

$$\mu'_\hbar = f_\hbar \mu_\hbar (f_\hbar^{-1} \otimes f_\hbar^{-1}) \quad (a) \quad (1-44)$$

$$\Delta'_\hbar = (f_\hbar \otimes f_\hbar) \Delta_\hbar f_\hbar^{-1} \quad (b)$$

working to first order, the equations become

$$\mu'_1(a_1 \otimes a_2) = \mu_1(a_1 \otimes a_2) + f_1(a_1, a_2) - a_1 f_1(a_2) - f_1(a_1) a_2 \quad (a)$$

$$\Delta'_1(a) = \Delta_1(a) - \Delta(f_1(a)) + (f_1 \otimes \text{id} + \text{id} \otimes f_1) \Delta(a) \quad (b) \quad (1-45)$$

definition 16.

A pair of k -module maps (μ_1, Δ_1) is called a deformation (mod \hbar^2) of A if it satisfies

$$\begin{aligned} \mu_1(a_1, a_2 \otimes a_3) + \mu_1(a_1 \otimes a_2) a_3 &= (1-46) \\ &= a_1 \mu_1(a_2 \otimes a_3) + \mu_1(a_1 \otimes a_2) a_3 \end{aligned}$$

$$\begin{aligned} (\Delta \otimes \text{id}) \Delta_1(a) + (\Delta_1 \otimes \text{id}) \Delta(a) &= (1-47) \\ &= (\text{id} \otimes \Delta) \Delta_1(a) + (\text{id} \otimes \Delta_1) \Delta(a) \end{aligned}$$

$$\begin{aligned} \Delta(\mu_1(a_1 \otimes a_2)) + \Delta_1(a_1, a_2) &= (1-48) \\ &= (\mu \otimes \mu_1 + \mu_1 \otimes \mu) \Delta^3(a_1) \Delta^2(a_2) + \\ &+ \Delta_1(a_1) \Delta(a_2) + \Delta(a_1) \Delta(a_2) \end{aligned}$$

definition 17

More generally, a $2n$ -tuple $(\mu_1, \dots, \mu_n, \Delta_1, \dots, \Delta_n)$ of k -module maps is a deformation (mod \hbar^{n+1}) of A if the relations:

$$\mu_h (\mu_h (a_1 \otimes a_2) \otimes a_3) = \mu_h (a_1 \otimes \mu_h (a_2 \otimes a_3)) \quad (1-49)$$

$$(\Delta_h \otimes \text{id}) \Delta_h (a) = (\text{id} \otimes \Delta_h) \Delta_h (a) \quad (1-50)$$

$$\Delta_h (\mu_h (a_1 \otimes a_2)) = (\mu_h \otimes \mu_h) \Delta_h^{13} (a_1) \Delta_h^{24} (a_2)$$

$$\text{hold } (\text{mod } h^{n+1}). \quad (1-51)$$

Notes

- Any deformation of A induces a deformation $(\text{mod } h^{n+1})$ for all n .
However, a deformation $(\text{mod } h^{n+1})$ does not in general extend to a genuine deformation.

- we write $\Delta_h^{13} (a_1)$ instead of

$$\Delta_h (a)_{13}$$

Remark 6

Twisting

Let $(A, \mu, \Delta, i, \varepsilon, S)$ be a Hopf algebra over a field K .

Let F be an invertible element of $A \otimes A$ such that

$$F_{12} (\Delta \otimes \text{id})(F) = F_{23} (\text{id} \otimes \Delta)(F) \quad (1-52)$$

$$(\varepsilon \otimes \text{id})(F) = 1 = (\text{id} \otimes \varepsilon)(F) \quad (1-53)$$

1. Then,

$$U = \mu (\text{id} \otimes S)(F) \quad (1-54)$$

is an invertible element of A with

$$U^{-1} = \mu (S \otimes \text{id})(F^{-1}) \quad (1-55)$$

2. If we define

$$\Delta^F : A \rightarrow A \otimes A$$

and $S^F : A \rightarrow A$

by

$$\Delta^F(a) = F \Delta(a) F^{-1} \quad (a) \quad (1-56)$$

$$S^F(a) = U S(a) U^{-1} \quad (b).$$

then $(A, \mu, i, \Delta^F, \varepsilon, S^F)$ is a Hopf algebra, denoted by A^F and called the twist of A by F .

3. we assume in addition that A is quasi-triangular with universal element R , and that

$$F_{21} = F^{-1} \quad \text{and}$$

$$F_{12} F_{13} F_{23} = F_{23} F_{13} F_{12} \quad (1-57)$$

Then A^F is quasi-triangular with universal element

$$R^F = F^{-1} R F^{-1} \quad (1-58)$$

Remark 7

If A is a cocommutative Hopf algebra and F is an invertible element of $A \otimes A$ satisfying (1-52), (1-53) then A^F is a triangular Hopf algebra with universal R element

$$R = F_2, F^{-1} \quad (1-59)$$

Note.

A is cocommutative if

$$\tau \circ \Delta = \Delta$$

(1-60)

we see that there is a way to construct quasitriangular Hopf algebras by starting with a cocommutative Hopf algebra A and "twisting" it with an element $F \in A \otimes A$ satisfying conditions (1-52), (1-53).

3.1 Quantized Universal Enveloping Algebras (Q.U.E.A)

Remark 8.

1. Let g be a Lie algebra over $k = \mathbb{R}$, or $k = \mathbb{C}$.

Let T be the (free) tensor algebra over g considered as a vector space i.e

$$T = \bigoplus_{z=0}^{\infty} T^z = k \oplus g \oplus (g \otimes g) \oplus (g \otimes g \otimes g) \oplus \dots \quad (1-61)$$

Let J be the two sided ideal in T generated by the elements of the form $x \otimes y - y \otimes x - [x, y]$, $\forall x, y \in g$ (1-62)

the quotient algebra

$$E = T/J \quad (1-63)$$

is called the universal enveloping algebra of L .

2. The ideal J in the definition has the form

$$J = \sum_{x, y \in g} T \otimes u_{xy} \otimes T \quad (1-64)$$

$$u_{xy} = x \otimes y - y \otimes x - [x, y] \quad (1-65)$$

$\forall x, y \in g$

3. The canonical map

$$\pi: T \rightarrow T/J = E$$

induces a linear map of \mathfrak{g} into E , we have

$$\forall x, y \in \mathfrak{g} \quad \pi[x, y] = [\pi(x), \pi(y)] \quad (1-66)$$

4. Let $\rho: \mathfrak{g} \rightarrow V$ be

a representation of a Lie algebra \mathfrak{g} in a vector space V , the formula

$$\rho(X_{i_1} \otimes X_{i_2} \otimes \dots \otimes X_{i_k}) = \rho(X_{i_1}) \dots \rho(X_{i_k}) \quad (1-67)$$

defines uniquely a representation $\hat{\rho}$ of the associative algebra $E(\mathfrak{g})$ in V .

(we write $T(\mathfrak{g}), E(\mathfrak{g})$ for T or E)
as in (1-61), (1-63)).

5. The P.B.W theorem asserts that

if $\{X_i\}_{i \in I}$ is any basis of \mathfrak{g} ,

where the index set I is totally ordered, the set of monomials

$$\{X_{i_1} X_{i_2} \dots X_{i_k}\} \text{ where } k \geq 1 \quad (1-68)$$

$i_1 \leq i_2 \leq \dots \leq i_k$ is a basis of $E(\mathfrak{g})$,

It follows that the composite of the natural map

$$g \rightarrow T(g) \text{ with the canonical}$$

$$\text{map } T(g) \rightarrow U(g)$$

gives an embedding of g into $U(g)$ (we write $U(g)$ for $E(g)$)

we usually identify g with its image under this map.

Remark 9.

Hopf structure on $U(g)$.

1. we define

$$\forall x \in g \quad \Delta(x) = x \otimes 1 + 1 \otimes x$$

$$S(x) = -x \quad (1-69)$$

$$\epsilon(x) = 0$$

Since g clearly generates $U(g)$ as an algebra, to define a Hopf algebra structure on $U(g)$ it is enough to give the structure maps on elements of g .

2. $U(g)$ is cocommutative, since $\Delta(x)$ is obviously contained in the symmetric part of $U(g) \otimes U(g)$, which is a subalgebra of $U(g) \otimes U(g)$.

definition 18

quantization

1. A Hopf algebra deformation of the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra \mathfrak{g} is called a quantized universal enveloping algebra (Q.U.E.A)
2. A Hopf algebra deformation of the algebra $F(G)$ of regular functions on an algebraic group G is called a quantized function algebra or simply a QF-algebra.

Note.

For a semisimple Lie algebra \mathfrak{g} , every deformation of $U(\mathfrak{g})$ is ~~isom~~ isomorphic to $U(\mathfrak{g})[[\hbar]]$ as an algebra.

Every deformation of $F(G)$ is isomorphic to $F(G)[[\hbar]]$ as a coalgebra

Remark 10

It is possible to characterize a semi-simple Lie algebra completely by giving the Lie brackets for the generators associated to simple roots in a Chevalley basis, together with the Jacobi identity and a few further relations:

The semi-simple Lie algebra \mathfrak{g} associated to a set

$$\Phi_S = \{ \alpha^{(i)}, i=1, \dots, r \}$$

of r simple roots $\alpha^{(i)}$ is uniquely determined as follows:

There are $3r$ generators

$\{ E_{\pm}^i, H^i : i=1, \dots, r \}$ with brackets

$$[H^i, H^j] = 0$$

$$[H^i, E_{\pm}^j] = \pm A^{ji} E_{\pm}^j \quad (\text{or}) \quad (1-70)$$

$$[E_{+}^i, E_{-}^j] = \delta^{ij} H^i$$

These generators obey the Jacobi identity

$$(\text{ad } E_{\pm}^i)^{1-A^{ji}} E_{\pm}^j = 0$$

for $i, j = 1, \dots, r, i \neq j$

Here $E^i = E^{\alpha^{(i)}}$, A is the $r \times r$

Cartan matrix associated to the simple roots ϕ_i and $(\text{ad}_x)^n$ is used as a shorthand notation for

$$\text{ad}_x \circ \text{ad}_x \circ \dots \circ \text{ad}_x,$$

so that e.g.

$$(\text{ad}_x)^2 \triangleq [x, [x, y]] \quad (1-71)$$

The Cartan matrix $A(g)$ of the semisimple Lie algebra g is the $r \times r$ matrix with elements

$$A^{ij}(g) = 2 \frac{(\alpha^{(i)}, \alpha^{(j)})}{(\alpha^{(j)}, \alpha^{(j)})} \quad (1-72)$$

for the simple roots.

For a simple Lie algebra g we have for the Cartan matrix

- (a) $A^{ii} = 2$
- (b) $A^{ij} = 0 \iff A^{ji} = 0$
- (c) $A^{ij} \in \mathbb{Z} \leq 0$, for $i \neq j$
- (d) $\det A > 0$
- (e) indecomposability

definition 19.

Consider a semisimple Lie algebra g . we construct the

Q.V.E.A $U_q(\mathfrak{g})$ as follows:

$U_q(\mathfrak{g})$ is the algebra of power series in the $3r+1$ generators

$$\{E_{\pm}^i, H^i : i=1, \dots, r\} \cup \{1\}$$

modulo the relations

$$(a) [H^i, H^j] = 0$$

$$(b) [H^i, E_{\pm}^j] = \pm A^{ji} E_{\pm}^j \tag{1-74}$$

$$(c) [E_+^i, E_-^j] = d^{ij} [H^i]$$

$$(c) \sum_{p=0}^{1-A^{ji}} (-1)^p \binom{1-A^{ji}}{p}_i (E_{\pm}^i)^p (E_{\pm}^j) (E_{\pm}^i)^{1-A^{ji}-p} = 0, \text{ for } i \neq j$$

$$(d) 1 \circ x = x = x \circ 1, \forall x \in U_q(\mathfrak{g})$$

with A^{ij} the Cartan integers of the Lie algebra \mathfrak{g} .

The square brackets are to be understood as commutators.

$$[x, y] \stackrel{\Delta}{=} x \circ y - y \circ x \tag{1-75}$$

with $m(x \otimes y) \stackrel{\Delta}{=} x \otimes y$

the formal product in $U(\mathfrak{g})$.

we have used the q -number symbol

$$[x] \stackrel{\Delta}{=} [x]_q = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}} \quad (1-76)$$

($\stackrel{\Delta}{=}$ means "we define... as")

together with

$$[x]_i \stackrel{\Delta}{=} [x]_{q_i} \quad \text{with} \quad (1-77)$$

$$q_i = q^{(\alpha^{(i)}, \alpha^{(i)}) / (\theta, \theta)} \quad (1-78)$$

$$[n]! = \prod_{m=1}^n [m] \quad (1-79)$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]! [n-m]!} \quad (1-80)$$

Notes:

- The θ is the highest root of \mathfrak{g}
 θ is the unique root such that
 $ht(\theta) > ht(\alpha)$, for all $\alpha \in \Phi \setminus \{\theta\}$
 (The height, ht of a root α of \mathfrak{g}
 is the sum of its components in the
 basis of simple roots
 $ht(\alpha) = \sum_{i=1}^r b_i$, for $\alpha = \sum_{i=1}^r b_i \alpha^{(i)}$ (1-81))

$$\forall \alpha \in \Phi \setminus \{\theta\} \quad ht(\theta) > ht(\alpha)$$

- For a semisimple Lie algebra \mathfrak{g} ,

the Weyl vector ρ is defined

$$a) \quad \rho \triangleq \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha \quad (1-82)$$

for $\xi \in \mathbb{C}$, we take as q ,

$$q = \exp \xi$$

and for any complex number μ we introduce the number

$$[\mu] \triangleq [\mu]_q = \frac{q^{\mu/2} - q^{-\mu/2}}{q^{1/2} - q^{-1/2}} \quad (1-82)$$

(and 1-76).

3. The exponential functions of generators appearing in the expressions $[H^i]$ in (1-74) (c) are defined through the corresponding power series i.e

$$e^{\xi H} = \sum_{n=0}^{\infty} \frac{\xi^n}{n} H^n \quad (1-83)$$

we define inductively

$$H^n \triangleq \underbrace{H_0 H_0 \dots H_0}_{n \text{ times}} \triangleq H^{0n} \quad (1-84)$$

In particular

$$e^{\xi H} \circ e^{-\xi H} = 1 \quad (1-85)$$