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Quantum Groups, Conformal  
field theory and noncommu-  
tative geometry.

by

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# First article

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Correlation functions for a WZW theory and representations of quasitriangular Hopf algebras - Part one

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## Abstract

For a finite dimensional complex simple Lie algebra  $\mathfrak{g}$  we consider the associated standard Hopf QVE algebra, we examine the equivalence according to the theory of Kohno and Drinfeld of the two representations of the braid group  $P_{KZ}$  and  $P_{AYBE}$ . We consider the KZ equation as derived in the WZW conformal field theory and in the affine Lie algebras theory.

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# 1. Braid group, braided vector space

## definition 1 (Artin's braid group)

Consider an integer  $n \geq 3$ . The braid group  $B_n$  on  $n$  strands, is the group with  $n-1$  generators  $\sigma_1, \dots, \sigma_{n-1}$  and relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i-j| > 1 \quad (a) \quad (7-1)$$

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{for } |i-j| = 1 \quad (b)$$

we can write

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } 1 \leq i \leq n-1 \quad (b)'$$

we define for  $n=2$  the braid group  $B_2$  as the free group with one generator and we let

$$B_0 = B_1 = \{1\}$$

be the trivial group. (1-2)

## definition 2

Let  $K$  be a field.

A braided vector space, is a vector space  $V$  together with an invertible  $K$ -linear map

$$C: V \otimes V \rightarrow V \otimes V$$

which obeys the equation:

$$\begin{aligned}
 (c \otimes id_V) \circ (id_V \otimes c) \circ (c \otimes id_V) &= \\
 = (id_V \otimes c) \circ (c \otimes id_V) \circ (id_V \otimes c) & \quad (1-3)
 \end{aligned}$$

in  $End(V \otimes V \otimes V)$   
 $id_V$  denotes the identity map  $V \rightarrow V$   
 $End(V \otimes V \otimes V)$  denotes the endomorphisms  $V \otimes V \otimes V \rightarrow V \otimes V \otimes V$ .

Corollary 1.

Let  $(v_i)_{i \in I}$  be a  $K$ -basis of  $V$ .  
 This allows us to describe  $c \in End(V \otimes V)$   
 by a family  $(c_{ij}^{kl})_{i,j,k,l \in I}$

of scalars:  $k, l$   
 $c(v_i \otimes v_j) = \sum_{k,l} c_{ij}^{kl} v_k \otimes v_l \quad (1-4)$

Corollary 2.

If  $c$  is invertible, then  $c$  describes  
 an isomorphism of vector spaces if and only  
 if, the following equation holds;

$$\sum_{p,q} c_{ij}^{pq} c_{qk}^{yn} c_{py}^{lm} = \sum_{y,q,r} c_{jk}^{qr} c_{iq}^{ly} c_{yr}^{mn} \quad (1-5)$$

$\forall l, m, n, i, j, k \in I$   
 $I$  is the set of indices.

### Corollary 3.

Let  $(V, c)$  be a braided vector space  
for  $1 \leq i \leq n-1$ , define an auto-  
morphism of  $V^{\otimes n}$ , by

$$C_i \stackrel{\Delta}{=} \left[ \begin{array}{ll} C \otimes \text{id}_{V^{\otimes (n-2)}} & \text{for } i=1 \\ \text{id}_{V^{\otimes (i-1)}} \otimes C \otimes \text{id}_{V^{\otimes (n-i-1)}} & \text{for } 1 < i < n-1 \\ \text{id}_{V^{\otimes (n-2)}} \otimes C & \text{for } i=n-1 \end{array} \right] \quad (1-5)$$

For any  $n > 0$  we have a unique  
homomorphism of groups

$$\rho_n^C : B_n \rightarrow \text{Aut}(V^{\otimes n}) \quad (1-6)$$

$$b_i \mapsto C_i \quad \text{for } i=1, 2, \dots, n-1 \quad (1-7)$$

Note

we write  $V^{\otimes n}$  for

$$\underbrace{V \otimes V \otimes \dots \otimes V}_{n \text{ - times}}$$

$\text{id}_{V^{\otimes n}}$  is the identity map for the  
space  $V^{\otimes n}$

$B_n$  is the braid group on  $n$ -strands.

Proof:

In fact the relations

$$C_i C_j = C_j C_i \text{ for } |i-j| \geq 2$$

holds, since the linear maps  $C_i, C_j$  act on different copies of the tensor product.

The relation

$$C_i C_{i+1} C_i = C_{i+1} C_i C_{i+1}$$

is part of the axioms of a braided vector spaces.



## 2. Quantum groups and representation of braid groups.

### Definition 3 Hopf algebra.

A Hopf algebra  $A$ , is a vector space endowed with five operations

$m: A \otimes A \rightarrow A$  multiplication

$u: F \rightarrow A$  unit map

$\Delta: A \rightarrow A \otimes A$  co-multiplication

$\epsilon: A \rightarrow F$  co-unit map

$S: A \rightarrow A$  antipode

which possess the following properties

1.  $m \circ (1_d \otimes m) = m \circ (m \otimes 1_d)$   
associativity

2.  $m \circ (1_d \otimes u) = m \circ (u \otimes 1_d)$   
existence of unit.

3.  $(1_d \otimes \Delta) \circ \Delta = (\Delta \otimes 1_d) \circ \Delta$   
co-associativity

4.  $(\epsilon \otimes 1_d) \circ \Delta = (1_d \otimes \epsilon) \circ \Delta = 1_d$   
existence of counity

5. connecting axiom

5.  $\Delta \circ m = (m \otimes m) \circ (\Delta \otimes \Delta)$
6. Existence of antipode

$$S: A \rightarrow A$$

$$m \circ (1_d \otimes S) \circ \Delta = \eta \circ \epsilon = m \circ (S \otimes 1_d) \circ \Delta$$

Note.

We consider  $A$  as a vector space over a zero characteristic field  $F$ .

$1_d$  is the identity  $A \rightarrow A$ .

Remark 1

For the multiplication  $m$  we have

$$m: A \otimes A \rightarrow A, \forall a, b \in A \quad m(a \otimes b) \stackrel{\Delta}{=} a \cdot b \in A$$

For the associativity we have

$$x \cdot m(y \cdot z) = (x \cdot y) \cdot z, \forall x, y, z \in A. \quad (1-8)$$

$\Delta$  means: we define as...

Remark 2

For the tensor product of two vector spaces  $A, B$  we have

for  $f: A \rightarrow A$ , or linear map,  
and  $g: B \rightarrow B$ , " " "

$$f \otimes g: A \otimes B \rightarrow A \otimes B$$

$$\forall a, b, a \in A, b \in B \quad (f \otimes g)(a \otimes b) =$$

$$= f(a) \otimes f(b) \quad (1-9)$$

For the algebra  $A$  we have

$$\forall x, y, z \in A \quad (Id \otimes m)(x \otimes y \otimes z) =$$

$$= x \otimes ymz$$

$$m \circ (Id \otimes m)(x \otimes y \otimes z) = m(x \otimes ymz) =$$

$$= x m(ymz) \quad \text{e.t.c.}$$

### Remark 3.

There is a nice notation for  $\Delta$ ,  
we write

$$\Delta(c) = \sum_i c_{i(1)} \otimes c_{i(2)} \quad (1-10)$$

$$\Delta: A \rightarrow A \otimes A$$

The right hand side is a formal sum denoting an element of  $A \otimes A$ .

we simply write

$$\Delta(c) = \sum c_{(1)} \otimes c_{(2)} \quad (1-11)$$

coassociativity then means that if we share out again, it does not matter which piece of  $\Delta(c)$  we share out.

Thus we write

$$\begin{aligned} c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} &= c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = \\ &= c_{(1)} \otimes c_{(2)} \otimes c_{(3)} \end{aligned} \quad (1-12)$$

we can write also, for the co-multiplication

$$\Delta: A \rightarrow A \otimes A$$

$$\forall x \in A \quad x \mapsto \Delta(x) = \sum_e x_1^{(e)} \otimes x_2^{(e)} \quad (1-13)$$

we have

$$\Delta(xm y) = \sum_{e, m} \left( x_1^{(e)} m y_1^{(m)} \right) \otimes \left( x_2^{(e)} m y_2^{(m)} \right) \quad (1-14)$$

The comultiplication and co-unit are homomorphisms of  $A$  i.e. they preserve the algebra multiplication

$$\Delta(xm y) = \Delta(x) m \Delta(y), \quad \forall x, y \in A$$

For the antipode we have

$$S(xm y) = S(y) m S(x), \quad \forall x, y \in A \quad (1-15)$$

that is the antipode  $S$  is an anti-homomorphism.

definition 4.

$$\pi: A \otimes A \rightarrow A \otimes A$$

$$\pi(x \otimes y) \stackrel{\Delta}{=} y \otimes x, \quad \forall x, y \in A \quad (1-16)$$

we say that  $\pi$  is a permutation

corollary 4:

The map

$$\Delta' = \pi \circ \Delta \quad (1-17)$$

is also a co-associative, co-multiplication.

- Contin.

Remark 4.

For a commutative algebra we have

$$m \circ \pi = m \tag{1-18}$$

a Hopf algebra  $A$ , analogously, is called co-commutative if

$$\pi \circ \Delta = \Delta \tag{1-19}$$

or in other words if  $\Delta'$  coincides with  $\Delta$ .

definition 5.

Let  $A$  be a Hopf algebra, an element  $R \in A \otimes A$  has an inverse if

$$R^{-1} m R = e \otimes e = R m R^{-1} \tag{1-20}$$

$e$  is the unit, we can extend the algebra multiplication  $m$ , we use the same symbol  $m$  for the multiplication in  $T(A)$ ,

$$T(A) = A \otimes \dots \otimes A, \text{ etc.}$$

definition 6.

Let  $A$  be a Hopf algebra, consider an element  $R \in A \otimes A$ , of the form

$$R = v_1 \otimes v_2, v_1, v_2 \in A \tag{1-21}$$

we define

$$R_{13} \in A \otimes A \otimes A, \quad R_{13} = r_1 \otimes e \otimes v_2$$

$$R_{12} \in A \otimes A \otimes A, \quad R_{12} = r_1 \otimes v_2 \otimes e \quad (1-22)$$

$$R_{23} \in A \otimes A \otimes A, \quad R_{23} = e \otimes r_1 \otimes v_2$$

$e$  is the unit in  $A$

### definition 7.

In the general case,  $R \in A \otimes A$   
and  $R \neq r_1 \otimes v_2$  we can write

$$R = \sum_e r_1^{(e)} \otimes v_2^{(e)}$$

and we can define similarly

$$R_{13} = \sum_e r_1^{(e)} \otimes e \otimes v_2^{(e)}, \quad R_{13} \in A \otimes A \otimes A$$

$$R_{12} = \sum_e r_1^{(e)} \otimes v_2^{(e)} \otimes e, \quad R_{12} \in A \otimes A \otimes A \quad (1-23)$$

$$R_{23} = \sum_e e \otimes r_1^{(e)} \otimes v_2^{(e)}, \quad R_{23} \in A \otimes A \otimes A.$$

### definition 8.

A quasi-triangular Hopf algebra  
is a pair  $(A, R)$  where  $A$  is a Hopf  
algebra and there exists:

$R \in A \otimes A$ ,  $R$  invertible with the  
properties

for  $\Delta' = \pi_0 \Delta$

$$\Delta'(x) = R_m \Delta(x) m R^{-1}, \quad \forall x \in A \quad (1-24)$$

i.e. the co-multiplications  $\Delta, \Delta'$  are related by conjugacies

$$(\text{id} \otimes \Delta)(R) = R_{13} m R_{12} \quad (a) \quad (1-25)$$

$$(\Delta \otimes \text{id})(R) = R_{13} m R_{23} \quad (b)$$

Note.

$$(\int \otimes \text{id}) R = R^{-1} \quad (1-26)$$

definition 9.

A quasi-triangular Hopf algebra is called triangular if and only if

$$R_{12} m R_{21} = e \otimes e \quad (1-27)$$

or in other words if and only if

$$\pi(R) = R^{-1}.$$

Remark 5.

we have the quantum Yang-Baxter equation (Q.Y.B.E)

$$R_{12} m R_{13} m R_{23} = R_{23} m R_{13} m R_{12} \quad (1-28)$$

proposition 7

Suppose that  $(H, R)$  is a quasi-triangular Hopf algebra, we define the operators

$$C_i: V^{\otimes m} \rightarrow V^{\otimes m}$$

by setting

$$C_i(u_1 \otimes \dots \otimes u_m) =$$

$$= u_1 \otimes \dots \otimes u_{i-1} \otimes \sigma(u_i \otimes u_{i+1}) \otimes u_{i+2} \otimes \dots \otimes u_m,$$

(1-29)

where  $\sigma \in \text{End}(V \otimes V)$  is the interchange of the two factors.

In this way we obtain a representation of a Braid group  $B_m$  on  $V^{\otimes m}$

$$u_1, \dots, u_m \in V, \quad V^{\otimes m} = \underbrace{V \otimes \dots \otimes V}_{m \text{ times}}$$

proof.

Indeed it is easy to check that the QYBE is equivalent to the braid group relations

$$C_i C_j C_i = C_j C_i C_j \quad \text{if } |i-j|=1 \quad (1-30)$$

$$\text{and } C_i C_j = C_j C_i, \text{ if } |i-j| > 1$$

This is obvious because if  $|i-j| > 1$



the operators  $\rho$   $C_i$  and  $C_j$  act on disjoint pairs of factors in  $V^{\otimes m}$ .

### definition 10

Consider the quasi-triangular Hopf algebra  $(V, R)$  and a braid group  $B_m$  on  $m$  strands and generators  $\tau_i$ .

We write

$$\rho^{QTB E}(\tau_i) = C_i \quad (1-31)$$

as defined in (1-29).

$\rho^{QTB E}$  is a representation of  $B_m$

### Note.

$V$  as a vector space, is a braided vector space.

We observe that given a quasi-triangular Hopf algebra  $(A, R)$  and a braid group on  $m$ -strands we construct the QTB E - representation:  $\rho^{QTB E}$ .

### 3. deformations of Hopf algebras

#### definition 11.

Let  $A$  be an associative algebra, a deformation of  $A$  is a family of bilinear associative maps

$$*_h : A \times A \rightarrow A$$

depending on a parameter  $h$ , such that

$$*_0 : A \times A \rightarrow A$$

is the given multiplication of  $A$ .

#### Note.

we consider only formal deformations, this means that we treat  $h$  as an indeterminate and expand  $*_h$  in a formal power series

$$\alpha_1 *_h \alpha_2 = \sum_{n=0}^{\infty} h^n \pi_n(\alpha_1, \alpha_2) \quad (1-32)$$

for certain bilinear maps

$$\pi_n : A \times A \rightarrow A$$

The associativity of  $*_h$  is equivalent to

$$\sum_{r+s=n} \pi_r(\pi_s(\alpha_1, \alpha_2), \alpha_3) = \quad (1-33)$$

$$= \sum_{r+j=n} \pi_r(\alpha_1, \pi_j(\alpha_2, \alpha_3))$$

for all  $\alpha_1, \alpha_2, \alpha_3 \in A$ ,  $n \geq 0$

### definition 12

1. Consider an algebra  $A$ . We write  $A[[h]]$  for the algebra of formal power series in  $h$  with coefficients in  $A$ .

That is

$$\alpha_h \in A[[h]] \quad (1-34)$$

$$\alpha_h = \alpha_0 + \alpha_1 h + \alpha_2 h^2 + \dots$$

for  $\alpha_0, \alpha_1, \alpha_2, \dots \in A$

2. Let  $K$  be a field of characteristic zero.

$K[[h]]$  is the ring of formal power series in an indeterminate  $h$  over the field  $K$ .

### definition 13

A topological Hopf algebra over  $K[[h]]$  is a complete  $K[[h]]$ -module  $A$ , equipped with  $K[[h]]$ -linear maps  $\mu, \Delta, i, \varepsilon, S$  satisfying the axioms for the definition of a Hopf algebra.

definition 14: deformation of a Hopf algebra

A deformation of a Hopf algebra  
 $(A, \mu, \Delta, i, \varepsilon, S)$  over a field  $K$  is  
 a topological Hopf algebra  $(A_h, \mu_h, \Delta_h,$   
 $i_h, \varepsilon_h, S_h)$   
 over the ring  $K[[h]]$  such that

1.  $A_h$  is isomorphic to  $A[[h]]$   
 as a  $K[[h]]$  module

2. For  $\alpha, \alpha' \in A$ , there are  $K$ -module  
 maps

$$\mu_h : A \otimes A \rightarrow A$$

$$\Delta_h : A \rightarrow A \otimes A$$

such that:

$$\begin{aligned} \mu_h(\alpha \otimes \alpha') &= \mu(\alpha \otimes \alpha') + \mu_1(\alpha \otimes \alpha')h + \\ &+ \mu_2(\alpha \otimes \alpha')h^2 + \dots \end{aligned} \quad (1-35)$$

$$\Delta_h(\alpha) = \Delta(\alpha) + \Delta_1(\alpha)h + \Delta_2(\alpha)h^2 + \dots$$

we write (1-36)

$$\mu_h \equiv \mu \pmod{h}, \quad \Delta_h \equiv \Delta \pmod{h}$$

definition 15:

Two deformations  $A_h, A'_h$  are

equivalent if there is an isomorphism

$$f_h : A_h \rightarrow A'_h$$

of Hopf algebras over  $K[[h]]$

which is the identity (mod  $h$ )

$$f_h = I \pmod{h} \quad (1-37)$$

that mean

$$f_h(\alpha) = \alpha + hf_1(\alpha) + h^2 f_2(\alpha) + \dots \quad (1-38)$$

$$f_i : A \rightarrow A$$

are isomorphisms of Hopf algebras

Note.

$$f_h^{-1}(\alpha) = \alpha - hf_1(\alpha) + \dots \quad (1-39)$$

$$\alpha \in A$$

Corollary 4

The associativity condition for  $\mu_h$  is

$$\begin{aligned} \mu_h(\mu_h(\alpha_1 \otimes \alpha_2) \otimes \alpha_3) &= \\ &= \mu_h(\alpha_1 \otimes \mu_h(\alpha_2 \otimes \alpha_3)) \end{aligned} \quad (1-40)$$

Similarly the coassociativity condition for  $\Delta_h$  is

$$(\Delta_h \otimes \text{id}) \Delta_h(\alpha) = (\text{id} \otimes \Delta_h) \Delta_h(\alpha) \quad (1-41)$$

$$\alpha_1, \alpha_2, \alpha \in A$$

writing both sides of this equation as formal power series in  $\hbar$ , we get an infinite number of conditions on the components  $\mu_i$  of  $\mu_\hbar$  and  $\Delta_i$  of  $\Delta_\hbar$ .

The components of order  $n$  gives

$$\begin{aligned} \mu_n(a_1 a_2 \otimes a_3) + \mu_n(a_1 \otimes a_2) a_3 &= \\ = a_1 \mu_n(a_2 \otimes a_3) + \mu_n(a_1 \otimes a_2 a_3) & \quad (1-42). \end{aligned}$$

and

$$\begin{aligned} (\Delta \otimes \text{id}) \Delta_1(a) + (\Delta_1 \otimes \text{id}) \Delta(a) &= \\ = (\text{id} \otimes \Delta) \Delta_1(a) + (\text{id} \otimes \Delta_1) \Delta(a) & \quad (1-43). \end{aligned}$$

we write  $a_1, a_2$  for  $\mu(a_1 \otimes a_2)$

### Corollary 5:

The conditions for a  $K[[\hbar]]$ -module isomorphism

$$f_\hbar: A_\hbar \rightarrow A'_\hbar$$

to be an equivalence of deformations are

$$\mu'_\hbar = f_\hbar \mu_\hbar (f_\hbar^{-1} \otimes f_\hbar^{-1}) \quad (a) \quad (1-44)$$

$$\Delta'_\hbar = (f_\hbar \otimes f_\hbar) \Delta_\hbar f_\hbar^{-1} \quad (b)$$

working to first order, the equations become

$$\mu'_1(a_1 \otimes a_2) = \mu_1(a_1 \otimes a_2) + f_1(a_1, a_2) - a_1 f_1(a_2) - f_1(a_1) a_2 \quad (a)$$

$$\Delta'_1(a) = \Delta_1(a) - \Delta(f_1(a)) + (f_1 \otimes \text{id} + \text{id} \otimes f_1) \Delta(a) \quad (b) \quad (1-45)$$

definition 16.

A pair of  $k$ -module maps  $(\mu_1, \Delta_1)$  is called a deformation (mod  $k^2$ ) of  $A$  if it satisfies

$$\begin{aligned} \mu_1(a_1, a_2 \otimes a_3) + \mu_1(a_1 \otimes a_2) a_3 &= (1-46) \\ &= a_1 \mu_1(a_2 \otimes a_3) + \mu_1(a_1 \otimes a_2) a_3 \end{aligned}$$

$$\begin{aligned} (\Delta \otimes \text{id}) \Delta_1(a) + (\Delta_1 \otimes \text{id}) \Delta(a) &= (1-47) \\ &= (\text{id} \otimes \Delta) \Delta_1(a) + (\text{id} \otimes \Delta_1) \Delta(a) \end{aligned}$$

$$\begin{aligned} \Delta(\mu_1(a_1 \otimes a_2)) + \Delta_1(a_1, a_2) &= (1-48) \\ &= (\mu \otimes \mu_1 + \mu_1 \otimes \mu) \Delta^3(a_1) \Delta^2(a_2) + \\ &+ \Delta_1(a_1) \Delta(a_2) + \Delta(a_1) \Delta_1(a_2) \end{aligned}$$

definition 17

More generally, a  $2n$ -tuple  $(\mu_1, \dots, \mu_n, \Delta_1, \dots, \Delta_n)$  of  $k$ -module maps is a deformation (mod  $k^{n+1}$ ) of  $A$  if the relations:

$$\mu_h (\mu_h (a_1 \otimes a_2) \otimes a_3) = \mu_h (a_1 \otimes \mu_h (a_2 \otimes a_3)) \quad (1-49)$$

$$(\Delta_h \otimes \text{id}) \Delta_h (a) = (\text{id} \otimes \Delta_h) \Delta_h (a) \quad (1-50)$$

$$\Delta_h (\mu_h (a_1 \otimes a_2)) = (\mu_h \otimes \mu_h) \Delta_h^{13} (a_1) \Delta_h^{24} (a_2)$$

$$\text{hold (mod } h^{n+1}). \quad (1-51)$$

### Notes

- Any deformation of  $A$  induces a deformation (mod  $h^{n+1}$ ) for all  $n$ .  
However, a deformation (mod  $h^{n+1}$ ) does not in general extend to a genuine deformation.

- we write  $\Delta_h^{13} (a_1)$  instead of

$$\Delta_h (a)_{13}$$

### Remark 6

#### Twisting

Let  $(A, \mu, \Delta, i, \varepsilon, S)$  be a Hopf algebra over a field  $K$ .

Let  $F$  be an invertible element of  $A \otimes A$  such that

$$F_{12} (\Delta \otimes \text{id})(F) = F_{23} (\text{id} \otimes \Delta)(F) \quad (1-52)$$



$$(\varepsilon \otimes \text{id})(F) = 1 = (\text{id} \otimes \varepsilon)(F) \quad (1-53)$$

1. Then,

$$U = \mu(\text{id} \otimes S)(F) \quad (1-54)$$

is an invertible element of  $A$  with

$$U^{-1} = \mu(S \otimes \text{id})(F^{-1}) \quad (1-55)$$

2. If we define

$$\Delta^F : A \rightarrow A \otimes A$$

and

$$S^F : A \rightarrow A$$

by

$$\Delta^F(a) = F \Delta(a) F^{-1} \quad (a) \quad (1-56)$$

$$S^F(a) = U S(a) U^{-1} \quad (b).$$

then  $(A, \mu, i, \Delta^F, \varepsilon, S^F)$  is a Hopf algebra, denoted by  $A^F$  and called the twist of  $A$  by  $F$ .

3. we assume in addition that  $A$  is quasi-triangular with universal element  $R$ , and that

$$F_{21} = F^{-1} \quad \text{and}$$

$$F_{12} F_{13} F_{23} = F_{23} F_{13} F_{12} \quad (1-57)$$

Then  $A^F$  is quasi-triangular with universal element

$$R^F = F^{-1} R F^{-1} \quad (1-58)$$

## Remark 7

If  $A$  is a cocommutative Hopf algebra and  $F$  is an invertible element of  $A \otimes A$  satisfying (1-52), (1-53) then  $A^F$  is a triangular Hopf algebra with universal  $R$  element

$$R = F_2, F^{-1} \quad (1-59)$$

### Note.

$A$  is cocommutative if

$$\tau \circ \Delta = \Delta$$

(1-60)

we see that there is a way to construct quasitriangular Hopf algebras by starting with a cocommutative Hopf algebra  $A$  and "twisting" it with an element  $F \in A \otimes A$  satisfying conditions (1-52), (1-53).

### 3.1 Quantized Universal Enveloping Algebras (Q.U.E.A)

#### Remark 8.

1. Let  $g$  be a Lie algebra over  $k = \mathbb{R}$ , or  $k = \mathbb{C}$ .

Let  $T$  be the (free) tensor algebra over  $g$  considered as a vector space i.e

$$T = \bigoplus_{z=0}^{\infty} T^z = k \oplus g \oplus (g \otimes g) \oplus (g \otimes g \otimes g) \oplus \dots \quad (1-61)$$

Let  $J$  be the two sided ideal in  $T$  generated by the elements of the form  $x \otimes y - y \otimes x - [x, y]$ ,  $\forall x, y \in g$  (1-62)

the quotient algebra

$$E = T/J \quad (1-63)$$

is called the universal enveloping algebra of  $L$ .

2. The ideal  $J$  in the definition has the form

$$J = \sum_{x, y \in g} T \otimes u_{xy} \otimes T \quad (1-64)$$

$$u_{xy} = x \otimes y - y \otimes x - [x, y] \quad (1-65)$$

$\forall x, y \in g$

3. The canonical map

$$\pi: T \rightarrow T/J = E$$

induces a linear map of  $\mathfrak{g}$  into  $E$ , we have

$$\forall x, y \in \mathfrak{g} \quad \pi[x, y] = [\pi(x), \pi(y)] \quad (1-66)$$

4. Let  $\rho: \mathfrak{g} \rightarrow V$  be

a representation of a Lie algebra  $\mathfrak{g}$  in a vector space  $V$ , the formula

$$\rho(X_{i_1} \otimes X_{i_2} \otimes \dots \otimes X_{i_k}) = \rho(X_{i_1}) \dots \rho(X_{i_k}) \quad (1-67)$$

defines uniquely a representation  $\hat{\rho}$  of the associative algebra  $E(\mathfrak{g})$  in  $V$ .

(we write  $T(\mathfrak{g}), E(\mathfrak{g})$  for  $T$  or  $E$ )  
as in (1-61), (1-63)).

5. The P.B.W theorem asserts that

if  $\{X_i\}_{i \in I}$  is any basis of  $\mathfrak{g}$ ,

where the index set  $I$  is totally ordered, the set of monomials

$$\{X_{i_1} X_{i_2} \dots X_{i_k}\} \text{ where } k \geq 1 \quad (1-68)$$

$i_1 \leq i_2 \leq \dots \leq i_k$  is a basis of  $\hat{E}(\mathfrak{g})$ ,

It follows that the composite of the natural map

$$g \rightarrow T(g) \text{ with the canonical}$$

$$\text{map } T(g) \rightarrow U(g)$$

gives an embedding of  $g$  into  $U(g)$   
(we write  $U(g)$  for  $E(g)$ )

we usually identify  $g$  with its image under this map.

Remark 9.

Hopf structure on  $U(g)$ .

1. we define

$$\forall x \in g \quad \Delta(x) = x \otimes 1 + 1 \otimes x$$

$$S(x) = -x \quad (1-69)$$

$$\epsilon(x) = 0$$

Since  $g$  clearly generates  $U(g)$  as an algebra, to define a Hopf algebra structure on  $U(g)$  it is enough to give the structure maps on elements of  $g$ .

2.  $U(g)$  is cocommutative, since  $\Delta(x)$  is obviously contained in the symmetric part of  $U(g) \otimes U(g)$ , which is a subalgebra of  $U(g) \otimes U(g)$ .

## definition 18

### quantization

1. A Hopf algebra deformation of the universal enveloping algebra  $U(\mathfrak{g})$  of a Lie algebra  $\mathfrak{g}$  is called a quantized universal enveloping algebra (Q.U.E.A)
2. A Hopf algebra deformation of the algebra  $F(G)$  of regular functions on an algebraic group  $G$  is called a quantized function algebra or simply a QF-algebra.

### Note.

For a semisimple Lie algebra  $\mathfrak{g}$ , every deformation of  $U(\mathfrak{g})$  is ~~isom~~ isomorphic to  $U(\mathfrak{g})[[\hbar]]$  as an algebra.

Every deformation of  $F(G)$  is isomorphic to  $F(G)[[\hbar]]$  as a coalgebra

### Remark 10

It is possible to characterize a semi-simple Lie algebra completely by giving the Lie brackets for the generators associated to simple roots in a Chevalley basis, together with the Jacobi identity and a few further relations:

The semi-simple Lie algebra  $\mathfrak{g}$  associated to a set

$$\Phi_S = \{ \alpha^{(i)}, i=1, \dots, r \}$$

of  $r$  simple roots  $\alpha^{(i)}$  is uniquely determined as follows:

There are  $3r$  generators

$\{ E_{\pm}^i, H^i : i=1, \dots, r \}$  with brackets

$$[H^i, H^j] = 0$$

$$[H^i, E_{\pm}^j] = \pm A^{ji} E_{\pm}^j \quad (\text{or}) \quad (1-70)$$

$$[E_{+}^i, E_{-}^j] = \delta^{ij} H^i$$

These generators obey the Jacobi identity

$$(\text{ad } E_{\pm}^i)^{1-A^{ji}} E_{\pm}^j = 0$$

for  $i, j = 1, \dots, r, i \neq j$

Here  $E^i = E^{\alpha^{(i)}}$ ,  $A$  is the  $r \times r$

Cartan matrix associated to the simple roots  $\phi_i$  and  $(\text{ad}_x)^n$  is used as a shorthand notation for

$$\text{ad}_x \circ \text{ad}_x \circ \dots \circ \text{ad}_x,$$

so that e.g.

$$(\text{ad}_x)^2 \triangleq [x, [x, y]] \quad (1-71)$$

The Cartan matrix  $A(g)$  of the semisimple Lie algebra  $g$  is the  $r \times r$  matrix with elements

$$A^{ij}(g) = 2 \frac{(\alpha^{(i)}, \alpha^{(j)})}{(\alpha^{(i)}, \alpha^{(i)})} \quad (1-72)$$

for the simple roots.

For a simple Lie algebra  $g$  we have for the Cartan matrix

- (a)  $A^{ii} = 2$
- (b)  $A^{ij} = 0 \Leftrightarrow A^{ji} = 0$
- (c)  $A^{ij} \in \mathbb{Z} \leq 0$ , for  $i \neq j$
- (d)  $\det A > 0$
- (e) indecomposability

### definition 19.

Consider a semisimple Lie algebra  $g$ . we construct the



Q.V.E.A  $U_q(\mathfrak{g})$  as follows:

$U_q(\mathfrak{g})$  is the algebra of power series in the  $3r+1$  generators

$$\{E_{\pm}^i, H^i : i=1, \dots, r\} \cup \{1\}$$

modulo the relations

(a)  $[H^i, H^j] = 0$

(b)  $[H^i, E_{\pm}^j] = \pm A^{ji} E_{\pm}^j$  (1-74)

(c)  $[E_+^i, E_-^j] = d^{ij} [H^i]$

(c)  $\sum_{p=0}^{1-A^{ji}} (-1)^p \begin{bmatrix} 1-A^{ji} \\ p \end{bmatrix}_i (E_{\pm}^i)^p (E_{\pm}^j) (E_{\pm}^i)^{1-A^{ji}-p} = 0$ , for  $i \neq j$

(d)  $1 \otimes x = x = x \otimes 1, \forall x \in U_q(\mathfrak{g})$

with  $A^{ij}$  the Cartan integers of the Lie algebra  $\mathfrak{g}$ .

The square brackets are to be understood as commutators.

$$[x, y] \triangleq x \circ y - y \circ x \quad (1-75)$$

with  $m(x \otimes y) \triangleq x \otimes y$

the formal product in  $U(\mathfrak{g})$ .

we have used the  $q$ -number symbol

$$[x] \stackrel{\Delta}{=} [x]_q = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}} \quad (1-76)$$

( $\stackrel{\Delta}{=}$  means "we define... as")

together with

$$[x]_i \stackrel{\Delta}{=} [x]_{q_i} \quad \text{with} \quad (1-77)$$

$$q_i = q^{(\alpha^{(i)}, \alpha^{(i)}) / (\theta, \theta)} \quad (1-78)$$

$$[n]! = \prod_{m=1}^n [m] \quad (1-79)$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]! [n-m]!} \quad (1-80)$$

Notes:

1. The  $\theta$  is the highest root of  $\mathfrak{g}$   
 $\theta$  is the unique root such that  
 $ht(\theta) > ht(\alpha)$ , for all  $\alpha \in \Phi \setminus \{\theta\}$   
 (The height,  $ht$  of a root  $\alpha$  of  $\mathfrak{g}$   
 is the sum of its components in the  
 basis of simple roots  
 $ht(\alpha) = \sum_{i=1}^r b_i$ , for  $\alpha = \sum_{i=1}^r b_i \alpha^{(i)}$  (1-81))

$$\forall \alpha \in \Phi \setminus \{\theta\} \quad ht(\theta) > ht(\alpha)$$

2. For a semisimple Lie algebra  $\mathfrak{g}$ ,

the Weyl vector  $\rho$  is defined

$$a) \quad \rho \triangleq \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha \quad (1-82)$$

for  $\xi \in \mathbb{C}$ , we take as  $q$ ,

$$q = \exp \xi$$

and for any complex number  $\mu$  we introduce the number

$$[ \mu ] \triangleq [ \mu ]_q = \frac{q^{\mu/2} - q^{-\mu/2}}{q^{1/2} - q^{-1/2}} \quad (1-82)$$

(and 1-76).

3. The exponential functions of generators appearing in the expressions  $[H^i]$  in (1-74) (c) are defined through the corresponding power series i.e

$$e^{\xi H} = \sum_{n=0}^{\infty} \frac{\xi^n}{n} H^n \quad (1-83)$$

we define inductively

$$H^n \triangleq \underbrace{H \circ H \circ \dots \circ H}_{n \text{ times}} \triangleq H^{0n} \quad (1-84)$$

In particular

$$e^{\xi H} \circ e^{-\xi H} = 1 \quad (1-85)$$