

# The quantum group structure of a conformal field theory (wzw)

## Abstract.

we present explicit free field realizations of simple affine current algebras.

we examine differential operator realizations of simple Lie algebras using Gauss decompositions.

we quantise the realizations by translating into free fields, furthermore free field realizations of screening currents of both kinds are discussed.

In appendices we explain in some detail the quantum group structure of conformal field theory using screened vertex operators.

## Contents

# 1. A digression on Lie algebras

Let  $g$  be a simple Lie algebra  
of  $\dim g = d$  and  
 $\text{rank } g = r$  (2-1)

$h$  is a Cartan subalgebra of  $g$ .  
we denote by  $\Delta_+$  the set of  
positive roots. we write

$\alpha > \beta$  if for two roots  $\alpha - \beta \in \Delta_+$   
 $\alpha > 0$  means  $\alpha \in \Delta_+$ .

The simple roots are  $\{\alpha_i\}_{i=1, \dots, r}$   
we denote by  $\theta$  the highest root.

The root dual to  $\alpha$  is

$$\alpha^\vee = \frac{2\alpha}{\alpha^2} \quad (2-2)$$

The only nonvanishing elements of  
the Cartan form are

$$K_{\alpha, -\alpha} = K(e_\alpha, f_\alpha) = \frac{2}{\alpha^2} \quad (2-3)$$

$$K_{\alpha_i, \alpha_j} = K(h_i, h_j) = G_{ij}$$

The Cartan matrix is

$$\begin{aligned} A_{ij} &= \alpha_i^\vee \cdot \alpha_j = (\alpha_i^\vee, \alpha_j) = \\ &= G_{ij} \alpha_j^2 / 2. \end{aligned} \quad (2-4)$$

### Remark 1.

we use the triangular decomposition

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+ \quad (2-5)$$

we denote the raising and lowering operators  $e_\alpha \in \mathfrak{g}_+$ ,  $f_\alpha \in \mathfrak{g}_-$

respectively with  $\alpha \in \Delta_+$  or  $\alpha \in \Delta_-$ . we denote by  $\lambda_\alpha$  an arbitrary Lie algebra element.

For simple roots we sometimes write

$$\begin{aligned} e_i &= e_{\alpha_i} \\ f_i &= f_{\alpha_i} \end{aligned} \quad (2-6)$$

The commutator relations for the 3r generators  $e_i, h_i, f_i$  are

$$[h_i, h_j] = 0, [e_i, f_j] = \delta_{ij} h_j \quad (2-7)$$

$$[h_i, e_j] = A_{ij} e_j, [h_i, f_j] = -A_{ij} f_j$$

In addition we have

$$\begin{aligned} (ad e_i)^{1-A_{ij}} e_j &= 0 \\ (ad f_i)^{1-A_{ij}} f_j &= 0 \end{aligned} \quad i \neq j \quad (2-8)$$

The Weyl vector is defined as

$$\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha \quad (2-9)$$

and satisfies

$$\rho \cdot \alpha_i^\vee = 1 \quad (2-10)$$

we use the convention

$$f_{-a, -b}^{-\sigma} = -f_{ab}^{\sigma} \quad (2-11)$$

Furthermore in the Cartan-Weyl basis we have

1.  $[e_a, f_a] = h_a = G^{ij} (\alpha_i^\vee, \alpha_j^\vee) h_j$
2.  $h_{a+b} = \frac{1}{(a+b)^2} (a^2 h_a + b^2 h_b)$
3.  $[h_i, e_a] = (\alpha_i^\vee, a) e_a \quad (2-12)$
4.  $[h_i, f_a] = -(\alpha_i^\vee, a) f_a$

The metric  $G_{ij}$  is related to the Cartan matrix as

$$A_{ij} = \alpha_i^\vee \cdot \alpha_j = (\alpha_i^\vee, \alpha_j) = G_{ij} \alpha_j^2 / 2 \quad (2-13)$$

while the Cartan-Killing form, denoted by  $K$  is

$$\text{tr}(j_a j_b) = K_{ab}, \quad (2-14)$$

$$-K(e_a, f_b) = k_{a,-b} = \frac{2}{a^2} \delta_{a,b}$$

$$K_{ij} = K(h_i, h_j) = G_{ij}$$

### Remark 2,

The Weyl vector

$$\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha \quad (2-15)$$

satisfies  $\rho \cdot \alpha_i^\vee = 1$

The Dynkin labels  $\Lambda_k$  of the weight  $\Lambda$  are defined by

$$\Lambda = \Lambda_k \Lambda^{(k)}, \quad \Lambda_k = (\alpha_k^\vee, \Lambda) \quad (2-16)$$

where

$$\{\Lambda^{(k)}\}_{k=1, \dots, r}$$

is the set of fundamental weights satisfying

$$(\alpha_i^\vee, \Lambda^{(k)}) = \delta_i^k.$$

## 2. The triangular coordinates.

### definition 1

we introduce the Lie algebra elements

$$e(x) = x^a e_a, \in \mathfrak{g}_+ \quad (1) \quad (2-17)$$

$$f(x) = x^a f_a, \in \mathfrak{g}_- \quad (2)$$

and the corresponding Lie group elements

$$g_+(x) = e^{e(x)} \quad (2-18)$$

$$g_-(x) = e^{f(x)}$$

### definition 2.

we introduce the matrix representation  $C(x)$  of  $e(x)$  in the adjoint representation

$$\begin{aligned} C_{\alpha}^b(x) &= C(x)_{\alpha}^b = (x^b C_b)_{\alpha}^b = \\ &= -x^b f_{ba} \end{aligned} \quad (2-19)$$

we use the following notation for the block matrix elements

$$C = \begin{pmatrix} C_+^- & 0 & 0 \\ C_0^+ & 0 & 0 \\ C_-^+ & C_-^0 & C_-^- \end{pmatrix} \quad (2-20)$$

In  $C_+^+$  both row and column indices are positive roots.

In  $C_-^0$  the row index is a negative root and the column index is a Cartan algebra index.

$C_+^+$  etc are matrices themselves.

Note,

We shall use repeatedly that

$C_{a,b}^b(x)$  vanishes unless  $a < b$

corresponding to  $C_+^+$  being upper triangular with zeros in the diagonal, similarly  $C_-^-$  is lower triangular.

### 3. The associated affine algebras.

#### Remark 3

For the associated affine algebras the operator product expansion, OPE, of the associated current is

$$J_a(z) J_b(w) = \frac{\kappa_{ab} K}{(z-w)^2} + \frac{f_{ab}^c J_c(w)}{z-w} \quad (2-21)$$

where regular terms have been omitted.  $\kappa$  is the central extension and  $k^V = \frac{2\kappa}{\theta^2}$  is the level.

In the mode expansion

$$J_a(z) = \sum_{-\infty}^{\infty} J_{a,n} z^{-n-1} \quad (2-22)$$

we use the identification

$$J_{a,0} \stackrel{\Delta}{=} J_a \quad (2-23)$$

#### Corollary 1

The Sugawara energy-momentum tensor is

$$T(z) = \frac{1}{\theta^2 (\kappa^V + h^V)} \kappa^{ab} : J_a J_b : (z) = \quad (2-24)$$

$$= \frac{1}{\epsilon} : \sum_{\alpha > 0} \frac{1}{\alpha^2} (E_{\alpha} F_{\alpha} + F_{\alpha} E_{\alpha}) + \frac{1}{2} (H, H) : (z)$$



where we have introduced the parameter

$$t = \frac{g^2}{2} (k^{\nu} + h^{\nu}) \quad (2-25)$$

$h^{\nu}$  is the dual Coxeter number.

This tensor has central charge

$$C = \frac{k^{\nu} d}{k^{\nu} + h^{\nu}} \quad (2-26)$$

#### Remark 4.

The standard free field construction

For every positive root  $\alpha$ , we introduce a pair of free bosonic fields

$b_{\alpha}$ ,  $\gamma^{\alpha}$  of conformal weights

$(1, 0)$  satisfying the OPE

$$b_{\alpha}(z) \gamma^{\beta}(w) = \frac{\delta_{\alpha}^{\beta}}{z-w} \quad (2-27)$$

The corresponding energy-momentum tensor is

$$T_{\text{free}} = : \partial \gamma^{\alpha} b_{\alpha} : \quad (2-28)$$

with central charge

$$C_{\text{free}} = d - r \quad (2-29)$$

we will understand "properly" repeated

root indices as in (2-28) to be summed over the positive roots.

### Remark 5.

For every Cartan index  $i=1, \dots, r$  we introduce a free scalar boson  $\varphi_i$  with contraction

$$\varphi_i(z) \varphi_j(w) = G_{ij} \ln(z-w) \quad (2-30)$$

The energy-momentum tensor is

$$T_\varphi = \frac{1}{2} : \partial \varphi \partial \varphi : - \frac{1}{\sqrt{\epsilon}} \varphi \cdot \partial^2 \varphi \quad (2-31)$$

and has central charge

$$C_\varphi = r - \frac{h^r d}{k^r + h^r} \quad (2-32)$$

( This follows from ~~Freudenthal~~

Freudenthal - de Vries formula

$$\rho^2 = h^r \theta^2 d / 24 \quad )$$

The total free field realization of the Sugawara energy-momentum tensor is

$$T = T_{\text{gh}} + T_\varphi \quad (2-33)$$

Corollary 2

The vertex operator

$$V_{\Lambda}(z) = : e^{i\sqrt{\alpha} \Lambda} \cdot \varphi(z) : \quad (2-34)$$

$$\Lambda \cdot \varphi(z) = \Lambda_i \varphi_j(z) G^{ij} \quad (2-35)$$

has conformal weight

$$h(V_{\Lambda}) = \frac{1}{2\alpha} (\Lambda, \Lambda + 2\rho) \quad (2-36)$$

It is also affine primary corresponding to highest weight  $\Lambda$ .

#### 4. Differential operator realization.

##### definition 3.

we set

$$Z = e^{x^\alpha} e_\alpha \quad (2-37)$$

we consider a lowest weight vector  $\langle \lambda |$ ,

$$\langle \lambda | f_\alpha = 0 \quad (2-38)$$

$$\begin{aligned} \langle \lambda | h_i = \langle \lambda | \lambda_i, \quad \lambda_i = \langle \lambda, h_i \rangle = \\ = \langle \lambda, \alpha_i^\vee \rangle \end{aligned}$$

##### Remark 6.

A differential operator realization of a simple Lie algebra  $\mathfrak{g}$  on the polynomial ring  $\mathbb{C}[x^\alpha]$  is given by the following right action

1.  $E_\alpha(x, 0) \langle \lambda | Z = \langle \lambda | Z e_\alpha$
2.  $H_i(x, 0, \lambda) \langle \lambda | Z = \langle \lambda | Z h_i$
3.  $F_\alpha(x, 0, \lambda) \langle \lambda | Z = \langle \lambda | Z f_\alpha$

There is an isotopic coordinate  $x^\alpha$  for every positive root  $\alpha > 0$  and for

brevity we sometimes write

$$x^i = x^{\alpha_i}$$

We argue that  $E_\alpha, H_i, F_\alpha$  indeed represent a realization of  $\mathfrak{g}$ .

Note:

The verification is based on the associativity

$$\begin{aligned} \langle \lambda | (z e^{s j_\alpha}) e^{t j_\beta} &= \\ = \langle \lambda | z (e^{s j_\alpha} e^{t j_\beta}) & \end{aligned} \quad (2-40)$$

where a comparison of terms linear in  $st$  gives the desired commutator of the differential operators

$$[j_\alpha, j_\beta] = f_{\alpha\beta}^c j_c$$

$j_\alpha, j_\beta$  are elements of the Lie algebra  $\mathfrak{g}$

As for the affine currents we use the common notation  $J_\alpha$  to denote the differential operators.

Remark 7

The Gauss decomposition.

For the element

$$\langle \lambda | Z e^{tJ_\alpha}$$

we have

$$\begin{aligned} 1. \langle \lambda | Z \exp(t e_\alpha) &= \langle \lambda | \exp(x^\sigma e_\gamma + \\ &+ t V_\alpha^b(x) e_\beta + o(t^2)) = \\ &= \langle \lambda | \exp(t V_\alpha^b(x) e_\beta + o(t^2)) Z \end{aligned} \quad (2-41)$$

$$\begin{aligned} 2. \langle \lambda | Z \exp(t h_i) &= \\ &= \langle \lambda | \exp(t h_i) \exp(x^\sigma e_\gamma + t V_i^b(x) e_\beta + \\ &+ o(t^2)) = \\ &= \langle \lambda | \exp(t (V_i^b(x) e_\beta + \lambda_i) + \\ &+ o(t^2)) Z \end{aligned} \quad (2-42)$$

$$\begin{aligned} 3. \langle \lambda | Z \exp(t f_\alpha) &= \\ &= \langle \lambda | \exp(t Q_{-\alpha}^{-b}(x) f_\beta + o(t^2)) \cdot \\ &\cdot \exp(t P_{-\alpha}^j(x) h_j + o(t^2)) \cdot \\ &\cdot \exp(x^\sigma e_\gamma + t V_{-\alpha}^b(x) e_\beta + o(t^2)) = \\ &= \langle \lambda | \exp(t (P_{-\alpha}^j(x) \lambda_j + V_{-\alpha}^b(x) e_\beta) + \\ &+ o(t^2)) Z \end{aligned} \quad (2-43)$$

hence:

1.  $E_\alpha(x, \rho) = V_\alpha^\rho(x) \partial_\rho$
  2.  $H_i(x, \rho, \lambda) = V_i^\rho(x) \partial_\rho + \lambda_i$
  3.  $F_\alpha(x, \rho, \lambda) = V_{-\alpha}^\rho(x) \partial_\rho + \rho_{-\alpha}^j(x) \lambda_j$
- (2-44)

The notation:

$$\partial_\rho = \partial_{x^\rho}$$

has been introduced.

Since  $E_\alpha(x, \rho)$  is independent of  $\lambda$  it may be defined through or Gauss decomposition alone

Note.

The Gauss decompositions rely on the Campbell-Baker-Hausdorff (C.B.H)

$$e^A e^{tB} = \exp \left\{ A + t \sum_{n \geq 0} \frac{B_n}{n!} (-\text{ad}_A)^n B + o(t^2) \right\}$$

where the coefficients  $B_n$  are the Bernoulli numbers

$$B(u) = \frac{u}{e^u - 1} = \sum_{n \geq 0} \frac{B_n}{n!} u^n$$

$$B(u) - B(-u) = -u, \quad B_{2m+1} = 0 \text{ for } m \geq 1$$

$$B_0 = 1, \quad B_1 = -1/2, \quad B_2 = 1/6, \dots$$

$$B^{-1}(u) = \frac{e^u - 1}{u} = \sum_{n \geq 1} \frac{u^{n-1}}{n!}$$

## 5. Wakimoto free field realization.

The free field realization is obtained from the differential operator realizations by the substitution

$$\begin{aligned} \partial_\alpha &\rightarrow b_\alpha(z) \\ X^\alpha &\rightarrow \gamma^\alpha(z) \end{aligned} \quad (2-45)$$

$$\lambda_i \rightarrow \sqrt{t} \partial \varphi_i(z)$$

and of subsequent normal ordering contribution or anomalous term

$$F_\alpha^{\text{anom}}(\gamma(z), \partial\gamma(z)) \quad (2-46)$$

to be added to the lowering part.

This term must have conformal dimension 1, and hence is bound to be of the form

$$F_\alpha^{\text{anom}}(\gamma(z), \partial\gamma(z)) = \partial\gamma^b(z) F_{\alpha b}(\gamma(z))$$

giving rise to the following form of the free field realization.

$$E_\alpha(z) = :V_\alpha^b(\gamma(z)) b_b(z):$$



$$H_i(z) = : V_i^b(\gamma(z)) \phi_b(z) : + \sqrt{t} \psi_i(z) \quad (2-47)$$

$$F_\alpha(z) = : V_{-\alpha}^b(\gamma(z)) \phi_b(z) : + \sqrt{t} \psi_j(z) \rho_{-\alpha}^j(\gamma(z)) + \psi \gamma^b(z) F_{\alpha_b}(\gamma(z))$$

$$\Delta(J_\alpha) = 1.$$

The normal ordering part for a simple root has been known as

$$\psi \gamma^b(z) F_{\alpha_b}(\gamma(z)) = \psi \gamma^{\alpha_i}(z) \left( \frac{k+t}{\alpha_i^2} - 1 \right) [z]. \quad (2-48)$$

## 6. Differential screening operators. Screening currents.

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### definition 4.

The differential screening operators are defined by

$$\begin{aligned} \exp\{-te_\alpha\} G_+(x) &= & (2-49) \\ &= \exp\{tS_\alpha(x, 0) + o(t^2)\} G_+(x) \end{aligned}$$

$$S_\alpha(x, 0) = S_\alpha^b(x) \circ_b \quad (2-50)$$

### Note

we have

$$S_\alpha(x, 0) = \tilde{E}_\alpha(-x, -0) \quad (2-51)$$

so that

$$S_\alpha^b(x) = -\left[B(-C(x))\right]_\alpha^b \quad (2-52)$$

$B$  are the Bernoulli numbers,  $C(x)$  is defined in (2-19).

The screening currents of the first kind are constructed using these polynomials as building blocks, for simple roots.

### definition 5.

For each simple root we define the screening currents of the first kind as

$$S_j(w) = : S_{\alpha_j}^{\alpha}(z(w)) b_{\alpha}(w) : : e^{-\frac{1}{\sqrt{E}} \alpha_j \cdot \varphi(w)} \quad (2-53)$$

$: :$  denote normal products of the fields.

$$\alpha_j \cdot \varphi(w) = \frac{\alpha_j^2}{2} \varphi_j(w) \quad (2-54)$$

Remark 8.

$$1. \quad E_{\alpha}(z) S_j(w) = 0 \quad (2-55)$$

$$2. \quad H_i(z) S_j(w) = 0$$

$$3. \quad F_{\alpha}(z) S_j(w) = \frac{\partial}{\partial w} \left( \frac{-z t / \alpha_j^2}{z-w} Q_{-\alpha}^{-\alpha_j}(z(w)) \right)$$

$$: e^{-\frac{1}{\sqrt{E}} \alpha_j \cdot \varphi(w)} : )$$

$$4. \quad T(z) S_j(w) = \frac{\partial}{\partial w} \left( \frac{1}{z-w} S_j(w) \right)$$

Note

The last OPE expresses that the screening currents have conformal weights

$$\Delta(S_j) = 1$$

definition 6

Screening currents of the second kind

$$\bar{S}_j(\omega) = \begin{pmatrix} S_{\alpha_j}^b(\gamma(\omega)) b_b(\omega) e^{\frac{1}{\sqrt{2}} \alpha_j \cdot \phi(\omega)} \\ \vdots \end{pmatrix} e^{-\frac{2t}{\alpha_j^2}} \quad (2-56)$$

Note.

we have the OPE.

$$E_\alpha(z) \bar{S}_j(\omega) = 0 \quad (2-57)$$

$$H_i(z) \bar{S}_j(\omega) = 0 \quad (2-58)$$

$$\begin{aligned} F_{\alpha_i}(z) \bar{S}_j(\omega) &= -\frac{2t}{\alpha_j^2} \delta_{ij} \frac{\partial}{\partial \omega} \left( \frac{1}{z-\omega} \begin{pmatrix} S_{\alpha_j}^b(\gamma(\omega)) b_b(\omega) \\ \vdots \end{pmatrix} \right) \\ &= \frac{-2t}{\alpha_j^2} \delta_{ij} \frac{\partial}{\partial \omega} \left( \frac{1}{z-\omega} \begin{pmatrix} S_{\alpha_j}^b(\gamma(\omega)) b_b(\omega) e^{\frac{1}{\sqrt{2}} \alpha_j \cdot \phi(\omega)} \\ \vdots \end{pmatrix} \right) \end{aligned} \quad (2-59)$$

$$T(z) \bar{S}_j(\omega) = \frac{2}{z-\omega} \left( \frac{1}{z-\omega} \bar{S}_j(\omega) \right) \quad (2-60)$$

$$\Delta(\bar{S}_j) = 1$$

A proof in detail, is given in [1].

# 7. The quantum group structure

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## definition 7.

we construct the screened vertex operators  $U_{\lambda, \mathbf{I}}(z)$ , where

$\mathbf{I} = \{i_1, \dots, i_n\}$  is an ordered set of labels

$$U_{\lambda, \mathbf{I}}(z) = \int_{C_1} dw_1 S_{i_1}(w_1) \dots \int_{C_n} dw_n S_{i_n}(w_n) V_{\lambda}(z) \quad (2-61)$$

we introduce for every  $z$  a vector space  $V_{\lambda, z}$  spanned by screened vertex operators  $U_{\lambda, \mathbf{I}}(z)$ .

The ordering may be chosen such that  $C_i$  lies inside  $C_j$  for  $i > j$ .

Let us denote such contours as standard contours around  $z$ .

### Note

The contours all start at infinity, encircle counter-clockwise the branch point  $z$  once, and return to infinity.

They do not intersect, nor do they cross the branch cut from  $z$  to infinity.

The vertex operator itself belongs to  $V_\lambda$ , namely

$$V_\lambda = U_{\lambda, \phi} \quad (2-62)$$

It is called the "highest weight vector" distinguished by the absence of contour structure.

### definition 8

we introduce the operator

$$K_i(U_{\lambda, I}(z)) = \quad (2-63)$$

$$= e^{i\alpha} \left( \sum_{i_\lambda \in I} R_{i_i} + R_{i\lambda} \right), U_{\lambda, I}(z)$$

For every screening current  $S_i$  we define the contour creating operator  $F_i$  as

$$F_i(U_{\lambda, I}(z)) = \int_C dw (S_i(w) U_{\lambda, I}(z)) \quad (2-64)$$

where  $C$  is a standard contour around  $z$  and encloses the ones in  $U_{\lambda, I}(z)$ .

### definition 9

choosing the branch cuts inherent in  $V_{\lambda_1}(z_1)$  and  $V_{\lambda_2}(z_2)$  non-intersecting, we may define the commultiplication  $\Delta(F_i)$  on  $V_{\lambda_1}(z_1) \otimes V_{\lambda_2}(z_2)$

by

$$\Delta(F_i) \left( U_{\lambda_1, I_1}(z_1) \otimes U_{\lambda_2, I_2}(z_2) \right) = \int_C dw S_i(w) U_{\lambda_1, I_1}(z_1) U_{\lambda_2, I_2}(z_2) \quad (2-65)$$

### corollary 3.

$$\begin{aligned} \Delta(F_i) \left( U_{\lambda_1, I_1}(z_1) \otimes U_{\lambda_2, I_2}(z_2) \right) &= \\ &= F_i \left( U_{\lambda_1, I_1}(z_1) \right) U_{\lambda_2, I_2}(z_2) + \quad (2-66) \\ &+ K_i^{-1} \left( U_{\lambda_1, I_1}(z_1) \right) F_i \left( U_{\lambda_2, I_2}(z_2) \right) \end{aligned}$$

### Remark 9

By an obvious contour deformation we may have for the contour  $C$

$$C = C_1 + C_2$$

where  $C_1$  ( $C_2$ ) is a standard contour

around  $z_1, z_2$ .

Thereby we performed the reordering in (2-66) which resulted in

$$\Delta(F_i) = F_i \otimes 1 + k_i^{-1} \otimes F_i \quad (2-67)$$

the co-multiplication of  $k_i$  is trivial.

$$\Delta(k_i) = k_i \otimes k_i \quad (2-68)$$

Next we want to introduce the raising operator  $E_i$  which should be a contour annihilating operator, in some space dual to  $F_i$ .

Recall the fundamental property of the screening currents

$$[L_{-1}, \int_C dw S(w)] = \int_C dw \partial_w S(w) \quad (2-69).$$

leaving only boundary terms

Definition 10:

we define implicitly the operator  $\hat{E}_i$  by

$$L_{-1} V_{\lambda, I}(z) = \left[ L_{-1}, \prod_{i \neq l} \int_{C_l} dw S_i(w_l) \right] \cdot V_{\lambda, I}(z) +$$



$$\begin{aligned}
& + \prod_{i \in I} \int_{C_i} dw_e \int_{i_e} (\omega_e) L_{-1} V_{\lambda} (z) \\
& = - \sum_{j \in I} (q_j - q_j^{-1}) \int_j (\infty) \hat{E}_j U_{\lambda, I} (z) \\
& + \prod_{i \in I} F_i L_{-1} V_{\lambda} (z) \quad (2-70)
\end{aligned}$$

We have introduced the deformation parameters

$$q_i = e^{i\pi \Omega_{ii} / 2} \quad (2-70)'$$

The factor  $(q_i - q_i^{-1})$  is for later convenience.

The last term is a screened descendant of  $V_{\lambda}$  and  $J$  denotes the total set of different screening currents.

### Remark 10

In [1] is given the result for the explicit expression for the action of  $\hat{E}_i$  on  $U_{\lambda, I} (z)$ . This computation is carried out only for all screenings being identical.

we find,  $n = \dim I$

$$\left[ L_{-1}, \prod_{\substack{i \in I \\ c_e}} \int_{c_e} dw_e S_{i_e}(w_e) \right] V_\lambda(z) =$$

$$= - \sum_{\substack{i \in I \\ e}} (q_{i_e} - q_{i_e}^{-1}) \int_{i_e} (\infty).$$

$$\frac{1 - e^{2\pi i (\sum_{j > i_e} \Omega_{i_e j} + \Omega_{i_e \lambda})}}{q_{i_e} - q_{i_e}^{-1}}$$

$$e^{i\pi \sum_{j > i_e} \Omega_{i_e j}} \prod_{\substack{i' \in I \setminus \{i_e\}}} \int_{c_{e'}} dw_{e'} S_{i'_e}(w_{e'}) V_\lambda(z) \quad (2-71)$$

from which we obtain

$$\hat{E}_i U_{\lambda, I} = \sum_{\substack{i' \in I, i' \sim i \\ e}} \frac{1 - e^{2\pi i (\sum_{j > i_e} \Omega_{i_e j} + \Omega_{i_e \lambda})}}{q_{i_e} - q_{i_e}^{-1}} \quad (2-72)$$

$$e^{i\pi \sum_{j > i_e} \Omega_{i_e j}} U_{\lambda, I \setminus \{i_e\}}$$

here we have used that

$$U_{\lambda, I \setminus \{i\}} = 0 \text{ for } i \notin I.$$

The restriction in summation means

$$e \sim i \text{ if } S_{i_e} = S_i$$

In the special case of only one type of screening current (2-72) reduces to

$$U_{2,n} = U_{2,1} \quad (2-73)$$

$$\hat{E}_i U_{2,n} = \frac{1 - q_i^{2(n-1) \text{ mod } 2}}{q_i - q_i^{-1}} [n]_{q_i} U_{2,n-1}$$

Here we have introduced

$$[ \alpha ]_q = \frac{1 - q^\alpha}{1 - q} \quad (2-74)$$

Remark 11. The quantum group structure.

From the trivial co-multiplication of  $L_{-1}$  it is not difficult to work out that

$$\Delta(\hat{E}_i) = \hat{E}_i \otimes 1 + K_i^{-1} \otimes \hat{E}_i \quad (2-75)$$

after redefining the raising operator

$$\text{by } E_i = K_i \hat{E}_i \quad (2-76)$$

the comultiplication reads

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i \quad (2-77)$$

From the following definitions

of an antipode  $\gamma$

$$\gamma(E_i) = -E_i k_i^{-1} \quad (a)$$

$$\gamma(k_i) = k_i^{-1} \quad (b) \quad (2-78)$$

$$\gamma(F_i) = k_i f_i \quad (c)$$

and co-unit  $\varepsilon$

$$\varepsilon(E_i) = 0 \quad (a) \quad (2-79)$$

$$\varepsilon(k_i) = 1 \quad (b)$$

$$\varepsilon(F_i) = 0 \quad (c)$$

It is straight forward to verify that the above is indeed a Hopf algebra, we call it  $A$ .

The final task is to implement quasi-triangularity in the Hopf algebra  $A$ , we define

$R = R_{\lambda_1, \lambda_2}$  to be the braiding matrix of two screened vertex operators

$$U_{\lambda_1, I_1}(z_1) U_{\lambda_2, I_2}(z_2) = \quad (2-90)$$

$$= \sum_{I'_1, I'_2} [R_{\lambda_1, \lambda_2}]_{I_1, I_2}^{I'_1, I'_2} U_{\lambda_2, I'_2}(z_2) U_{\lambda_1, I'_1}(z_1)$$

Let us summarize the commutation relations which are not difficult to determine

1.  $k_i k_j = k_j k_i$
2.  $k_i E_j = e^{i n_{ij}} E_j k_i \quad (2-81)$
3.  $k_i F_j = e^{-i n_{ij}} F_j k_i$
4.  $[E_i, F_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}$

Remark 12:

we review the defining (Chevalley) commutation relations for the  $q$ -deformed enveloping algebra  $U_q(\mathfrak{g})$  of a simple Lie algebra  $\mathfrak{g}$ .

1.  $k_i k_j = k_j k_i$
2.  $k_i e_j = q_i^{A_{ij}} e_j k_i$
3.  $k_i f_j = q_i^{-A_{ij}} f_j k_i$
4.  $[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}$

where

$$q_i = q^{D_i}$$

$$D = \text{diag} (D_i)$$

is a diagonal matrix symmetrizing the Cartan matrix.

$$Q_i A_{ij} = Q_j A_{ji} \quad (2-83)$$

### Remark 13

In our case we find the deformation parameters rel. (2-70)' to be

$$q_j = e^{i\pi Q_{ij}/2} = \begin{cases} e^{i\pi \alpha_j^2 / 2t} & j \leq r \\ e^{i\pi (\alpha_{j-r}^\vee)^2 t / 2} & j > r \end{cases} \quad (2-84)$$

As immediately see that each screening current  $S_i$ ,  $i=1, \dots, 2r$  gives rise to a subalgebra  $U_{q_i}(sl(r))$  generated by  $\{E_i, K_i, F_i\}$  with  $q_i$  given by (2-84).

Thus the total quantum group  $A$  may be viewed as a semidirect sum of these  $2r$  subalgebras.

### Remark 14

These are not the only subalgebras. Both kinds of screening currents also

give rise to a subalgebra  
each.

we observe that with

$$q = e^{i\pi/t}, \quad \mathcal{D}_i = \alpha_i^2/2 \quad (2-85)$$

the screening currents of the first  
kind  $S_i, i \leq r$  give rise to the sub-  
algebra  $U_q(\mathfrak{g})$  due to

$$(q^{\mathcal{D}_i})^{\pm A_{ij}} = e^{\pm i\pi \Omega_{ij}} \quad (2-86)$$

similarly the screening currents of  
the second kind

$$S_i, i > r \quad \text{with}$$

$$\hat{q} = e^{i\pi t}, \quad \hat{\mathcal{D}}_i = \left( \alpha_{i-r}^v \right)^2 / 2 =$$

$$= 2 / \alpha_{i-r}^2 \quad (2-87)$$

give rise to the subalgebra  $U_{\hat{q}}(\mathfrak{g}^t)$

due to

$$(\hat{q}^{\hat{\mathcal{D}}_i})^{\pm A_{j-r, i-r}} = e^{\pm i\pi \Omega_{ij}} \quad (2-88)$$

Here  $\mathfrak{g}^t$  is the dual Lie algebra  
to  $\mathfrak{g}$ , obtained by transposing the Cartan  
matrix

$$A_{ij} \rightarrow A_{it}^t = A_{ji} \quad (2-89)$$

Hence alternatively we may view  $A$  as the semi-direct sum

$$A = \mathcal{U}_q(\mathfrak{g}) \oplus_{\text{semi}} \mathcal{U}_{\bar{q}}(\mathfrak{g}^t) \quad (2-90).$$

That the sum is only semi-direct is due to the fact that  $\Omega_{ij}$  is not a block matrix.

For admissible representations  $t$  is rational and the deformation parameters  $q$  and  $\bar{q}$  are roots of unity.

### Note.

We may generalise the construction above from vertex operators of weight  $\lambda$  to primary fields labelled by  $\lambda$ .

### Remark 15

According to Gomez and Sierra [6], we have the derivation of the quantum group structure in the general setting of

$$S_i(z) S_j(w) = e^{i\pi \Omega_{ij}} S_j(w) S_i(z) \quad (2-90)$$

$$S_i(z) V_\lambda(w) = e^{i\pi \Omega_{ij}} V_\lambda(w) S_i(z)$$



$$V_{\lambda}(z) V_{\lambda'}(w) = e^{i\pi \Omega_{\lambda\lambda'}} V_{\lambda'}(w) V_{\lambda}(z)$$

where  $\{S_i\}_{i=1, \dots, 2r}$  denote the set of screening currents and  $\{V_{\lambda}\}$  denote the set of vertex operators. we use the notation

$$S_i = s_i, \text{ and}$$

$$S_{i+r} = \tilde{s}_i \quad \text{for } 1 \leq i \leq r$$

According to [6] there exists a quantum group structure when (2-90) is satisfied for  $\Omega_{ij}$  and  $\Omega_{\lambda\lambda'}$  being symmetric.

Since there is no relative braiding between  $b$  and  $\gamma$  we get

$$b(z)^a \gamma(w)^b = \gamma(w)^b b(z)^a$$

hence the braiding commutation of the screening currents will be independent of the  $b, \gamma$  part.

Therefore in our case, we may easily deduce that:

$$R_{\lambda\lambda'} = \frac{1}{\epsilon} \lambda \cdot \lambda'$$

$$R_{i\lambda} = \begin{cases} -\alpha_i \cdot \lambda / \epsilon \\ \alpha_{i-r}^v \cdot \lambda \end{cases}$$

$$\left. \begin{array}{l} i \leq r \\ i > r \end{array} \right\} \text{(2-91)}$$

$$R_{ij} = \begin{array}{l} \alpha_i \cdot \alpha_j / \epsilon \\ -\alpha_i \cdot \alpha_{j-r}^v \\ -\alpha_{i-r}^v \cdot \alpha_j \end{array}$$

$$i, j \leq r$$

$$i \leq r < j$$

$$j \leq r < i$$

# Appendix 1

## Remark 16.

A free field realization of WZW theories is provided by the Wakimoto construction.

We express the currents and the primary fields of a WZW theory based on a simple algebra  $\bar{g}$  in terms of free bosons  $\varphi^i(z)$ ,

$$i = 1, \dots, r = \text{rank } \bar{g}$$

and of  $\frac{d-r}{2}$  pairs  $(\beta^\alpha, \gamma^\alpha)$

of (bosonic) ghosts where

$$d = \dim \bar{g}$$

i.e. there is one pair for each positive root  $\alpha$  of  $\bar{g}$ .

The construction of bosons fields has to be taken as (2-92)

$$\underbrace{\varphi_i(z) \varphi_j(w)} = -G_{ij} \ln(z-w)$$

where  $G_{ij}$  is the quadratic form matrix of  $\bar{g}$ , which together

