Nikolaos Konstantopoulos

Mathematical Diary
1993–1998

Fermat's Last Theorem
— van der Waerden's Theorem

Athens 2001

E-mail: nkonst@tee.gr

9 Omonoias St.
16342 Ilioupolis
Greece

A note concerning the written-by-hand text.

In the written-by-hand text we use the following conventions. A piece of text underlined by a wavy line, in printed form should be in bold face style. Also a piece of text underlined by a straight line, in printed form must be in italics. In some special pieces of text (theorems, definitions, etc.) instead of underlining by a straight line to denote the italic style, we prefer to place a bracket at the margin of the emphasized piece of text to avoid disturbing the formulae. So a piece of text indicated by a bracket at its margin is equivalent to being underlined by a straight line and in printed form must be in italics.

The main part of the book (Chapters 1–7), as we see at the beginning of Chapter 1, starts from page 7 instead of page 1, that is, pages 1–6 do not exist. This is due to misprogramming and, in fact, no text is missing.

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Introduction.

This work, as its title "Mathematical Diary 1993-1998" suggests, is the account of an investigation performed between the years 1993 and 1998, concerning Fermat's Last Theorem (F.L.Th.) and van der Waerden's theorem (vdW's theorem), which is presented in chronological order. The material in the chapters essentially is given in "raw form", as it has been produced, without a complete mathematical elaboration. The whole work consists of two parts clearly distinguished with respect to subject and style: Chapters 3 and 4 refer to F.L.Th. whereas Chapters 1,2 and 5,6,7 refer to vdW's theorem. As noted at the end of each chapter, the material was selected and written long enough after the actual time of its production. Nearly everywhere we follow the "chronological" style of presentation (everything is placed according to when it was produced) but in some cases (various parts of Chapter 1, §5 of Chapter 2, Procedure 16 and Proof 17 of Chapter 6) we have preferred to place the material according to its natural context.

Let us see, in brief, the content of each chapter. We consider as first part of the book Chapters 1 and 2, as second part Chapters 3 and 4, and as third part Chapters 5,6 and 7.

In the first part (Chapters 1,2) we present some elementary aspects of Ramsey theory, especially van der Waerden's theorem, and we attempt to find better upper bounds for the van der Waerden numbers (vdW numbers).

In Chapter 1 we give an introduction of Ramsey theory. We consider only the aspects of the theory that offered us the initial motivation for doing the investigation described in the next chapters (Chapters 2,5,6,7). The presentation is based to a great extent on Ref. 1, but also on Ref. 2. Analytically, we discuss: Ramsey's theorem, Ramsey numbers and their upper bounds, some other Ramseyian theorems, van der Waerden's theorem, the enormous upper bounds of the van der Waerden numbers which are described with the help of the Ackermann hierarchy.

In Chapter 2 our goal is to find better upper bounds for the vdW numbers. Once we have stated vdW's theorem the material here is practically independent of Ch. 1. In §1 we cite some remarks concerning vdW's theorem which are never mentioned again in the following. In §2 we conjecture, through examples, possible upper bounds for some special cases of vdW numbers. In §3 we attempt to generalize these conjectured possible upper bounds for any case of vdW numbers (we consider only two "colours" or "partitions"). In §4 we discuss some points of §2 and §3. Finally, trying to prove or disprove the conjectured upper bounds, we consider in §5 a different but related problem.

During its development Ramsey theory has been connected, somehow, to <u>Fermat's Last Theorem</u>. In Chapters 3 and 4 (second part of the book), which are totally independent of the previous Chapters 1 and 2, we attempt to find a proof of F.L.Th. using only elementary methods. This goal here has not been completed so we present only ideas and directions of thought for further investigation. The basic part of this work was performed when a proof for F.L.Th. (the proof found in the years 1993–1994) already existed. We insisted to do the work since we think that a theorem like this it is good to have more than one proofs. During the investigation, wishing to avoid being influenced by the many previous efforts (the efforts to be F.L.Th. proved, made in the past years and centuries), we have chosen consciously to remain unaware of these efforts. So, our attempt to prove F.L.Th. starts "from the zero". Any material in this work, that already exists in the literature, is just rediscovered here! In fact, these two chapters have strongly the style of a personal notebook: there are repetitions, parts unconnected or superfluous, auxiliary computations or examples, incomplete ideas and thoughts, incomplete results, etc. Perhaps some parts are useful only to the author himself. Many sections can be read independently.

In Chapter 3 we attempt to prove F.L.Th. by properly analysing the powers of the integers. The content of each section is denoted clearly in its title. §§1,2,3,4,5,6 can be read independently, whereas §§7,8,9,10,11,12 are interdependent and also dependent on §§5,6. The more basic results can be found in §§5,6,7 and especially in §§9,10. In fact, <u>the basic direction of work</u> for proving F.L.Th., suggested in this chapter, is described in §§9,10 and particularly in Tables 25, 26, 29, 30, 30a, 30b where we try to give the "architecture" of the theorem. In §12 essentially we have the transition from Chapter 3 to Chapter 4.

In Chapter 4, which is totally independent of Chapter 3, we attempt to prove F.L.Th. by using proper divisions of the two parts of equation

$x^n+y^n=z^n$. Wishing through these divisions to reveal the "incompatibility" between the two parts of the equation we examine various related topics. Ideas concerning these topics are gradually developed in the succession of the sections until in §§ 9–13, which we consider as the most essential part of the chapter, an interesting perspective appears with respect to the possibility of proving F.L.Th. by such methods. §§ 1–8 are interrelated but nearly each of them can, to a great extent, be read independently. There, for the most part, we concentrate mainly on the various topics whereas F.L.Th. itself is only implicitly present. §§ 9–13, which are the most important, are strongly interconnected but they can practically be read independently of §§ 1–8 (perhaps § 11 is not necessary for understanding §§ 12, 13). Here, we concentrate mainly on F.L.Th. itself. <u>The basic ideas are exposed in § 9, § 10 but also in § 12.</u> There, F.L.Th. essentially is reduced to problems of distribution of primes; also, F.L.Th. is connected to possible generalizations of Fermat's Little Theorem. The whole work of §§ 9–13 is based on Table 21; also, Table 22 is of special importance. Some of the ideas, developed in § 9 and § 10, are visually represented in § 12. In § 13 we note some interesting patterns.

Chapters 5, 6 and 7 (third part of the book), in which we consider again van der Waerden's theorem, are totally independent of Chapters 3 and 4 and practically they can be read independently also of the Chapters 1 and 2. Wishing to prove (or to disprove) the conjectured results of Chapter 2 we gradually have come to the direction of thought exposed in Chapter 5 (in fact §5 of Ch. 2, which "by context" has been placed in Chapter 2, "chronologically" should be placed between Chapter 4 and Chapter 5 in this "Mathematical Diary"). Then, recognizing that Chapter 5 essentially had opened an autonomous interesting perspective for studying vdW's theorem, we abandoned the questions of Chapter 2 to investigate independently this perspective. In Chapters 6 and 7 we examine, on an elementary level, some of the computational possibilities offered by the work of Chapter 5.

In Chapter 5 we give a detailed description of some other representation of van der Waerden's theorem. §1 contains preliminary things. The basic idea, on which the new representation of vdW's theorem has been founded, is presented in §2 and §3. The idea schematically is elucidated in Figure 1 whereas the basic results, obtained from this figure, are compactly given in Table 5. In §4 we show how Table 5 can be produced from Figure 1 in a more direct way. In §5 we consider vdW's theorem with the help of the "machinery" created in §2 and §3. In Figure 2 we elucidate pictorially how the most elementary non-trivial of the vdW numbers, namely number W(3), is formed in the new representation of vdW's theorem. Finally, in §6 we discuss the above results and the perspectives opened by them.

An important thing here is that perhaps analogous representations could be given for the other Ramseyian theorems as well. Such a fact could possibly lead to <u>a unified formulation of these theorems</u>. Ramseyian properties seem to be of fundamental significance in mathematics and in nature (some aspects of them remind us fractals, phase transitions, general classification of structures, etc.) so a better understanding of them from various points of view would be welcome. Considering here vdW's theorem as a "representative" of the above theorems we wish, with the help of the results of Ch. 5, <u>to investigate the structure of the vdW numbers</u>. As a basic example we use number W(3) [in Ch. 5 vdW's theorem is (considered only for) two "partitions"] expecting that the possible results about it will be, in some way, generalized for the other vdW numbers. For this purpose, we want to find a systematical procedure (an algorithm) for computing the value of W(3) with the help of Figure 2 of Ch. 5. Thus in Chapters 6 and 7 we perform auxiliary computations on Fig. 2 of Ch. 5. Despite the great amount of work needed in these two chapters and the lengthy descriptions, the produced algorithms are relatively simple and this is encouraging.

In Chapter 6, using number W(3) as model for the vdW numbers, we perform computations on Fig. 2 of Ch. 5 in a specific manner. In §1 we present some computations of this form to become familiar with their style. In §§2, 3, 4 we analyse through examples the computations and we find algorithms to perform them easily. In §5 the results (algorithms) of §§2, 3, 4 are written for the general case and we offer a proof of them. The basic results of the whole chapter are presented in Note 18 whereas the basic procedure for their production is given compactly in Tables 7 and 8. Finally in the Appendix of the chapter we present some thoughts about how the above computations could be used for determining the way by which number W(3) is produced [hence for computing W(3) as well].

In Chapter 7 we perform computations on Fig. 2 of Ch. 5 in a different manner than in Ch. 6 [wanting again to use them for revealing how number W(3) is obtained and which is its value — a goal not accomplished yet in this work]. In §1 we give definitions, notation and examples concerning the specific kind of computations. Also we discuss some features of these computations. In §2 wishing to find out how the computations are performed we examine some special cases of them which are shown in Table 3. In fact, the whole chapter is based on this table. In §2, also, we describe analytically the table, how it was produced, and what patterns appear in it. In §3 from the special cases of Table 3 we deduce the algorithm for doing the computations in the general case. This algorithm, which together with the procedure in Table 3 is the basic result of this chapter, is presented in Tables 5, 6 and compactly in Tables 8, 9a, 9b. Finally, in §4 we give the proof of this algorithm as well as the proof that the procedure in Table 3 is valid for the general case. This is accomplished by writing this procedure for the general case. To facilitate the whole work of §4 we introduce some additional compact notation.

Athens, March–April 2001.

N. K.